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ARTIN AND DEHN TWIST SUBGROUPS OF THE MAPPING CLASS GROUP

By

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To my parents

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TABLE OF CONTENTS

LIST OF FIGURES

ABSTRACT

This dissertation investigates two types of subgroups in the mapping class group of an orientable surface. The first type of subgroups are isomorphic images of Artin groups. The second type of subgroups is one which is generated by three Dehn twists along simple closed curves with small geometric intersections.

Let S be a compact orientable surface. The mapping class group, $Mod(S)$, of S is the group of isotopy classes of orientation preserving homeomorphisms of S fixing the boundary pointwise. $Mod(S)$ is a very rich and complex object. In this dissertation, we make progress toward understanding the structure of the above mentioned subgroups of $Mod(S)$.

We tackle three problems. The first problem focuses on finding embeddings of Artin groups into $Mod(S)$. The second problem involves finding Artin relations of every length in Mod(S). And the third problem deals with understanding subgroups of $Mod(S)$ generated by three Dehn twists along curves with small geometric intersections.

While it is easy to find nontrivial homomorphisms of Artin groups into $Mod(S)$, the question of whether such homomorphisms are injective is quite hard. In this dissertation, we find embeddings of the Artin groups $\mathcal{A}(B_n)$, $\mathcal{A}(H_3)$, $\mathcal{A}(I_2(n))$, and most notably $\mathcal{A}(\widetilde{A}_{n-1})$ into $Mod(S)$. Further, we prove that if a collection $\{a_1, \cdots, a_n\}$ of simple closed curves in S has curve graph (see definition 4.1.2) $\widetilde{A}_{n-1}$ and N_ϵ is a closed regular neighborhood of $\cup_{i=1}^{n} a_i$, then the subgroup of $Mod(N_\epsilon)$ generated by the (left) Dehn twists T_i along a_i is isomorphic to $\mathcal{A}(\widetilde{A}_{n-1})$ almost all the time.

In the second problem, we study Artin relations in the mapping class group. If $l \geq 2$ is an integer, then a and b satisfy the Artin relation of length l if $aba\cdots = bab\cdots$, where

each side of the equality has l terms. We give explicit elements of $Mod(S)$ satisfying Artin relations of every integer length $l \geq 2$. By direct computations, we find elements x and y in $Mod(S)$ satisfying Artin relations of every even length ≥ 8 and every odd length ≥ 3. Then using the theory of Artin groups, we give two methods for finding Artin relations in $Mod(S)$. The first yields Artin relations of every length ≥ 3, while the second provides Artin relations of every even length ≥ 6. In the last two cases, we also show that x and y generate the Artin group $\mathcal{A}(I_2(l))$, where l is the length of the Artin relation satisfied by x and y.

The third problem is concerned with understanding subgroups in $Mod(S)$ generated by three Dehn twists along curves with small geometric intersections. Let a_1, a_2, and a_3 be distinct isotopy classes of essential simple closed curves in an orientable surface S. Assume that $i(a_j, a_k) \in \{0, 1, 2\}$ for all j, k. Denote by T_i the (left) Dehn twist along a_i, and let G represent the subgroup of $Mod(S)$ generated by T_1, T_2, and T_3. Set $(x_{12}, x_{13}, x_{23}) = (i(a_1, a_2), i(a_1, a_3), i(a_2, a_3))$. We find explicit presentations for G when $(x_{12}, x_{13}, x_{23}) = (0, 0, 0)$, $(1, 0, 0)$, $(2, 0, 0)$, $(1, 0, 1)$, and $(1, 1, 1)$. For the triple $(2, 1, 0)$, there are two cases to consider (see subsections 7.8.1 and 7.8.2). In both cases, we are not able to find an explicit presentation for G. Nevertheless, we prove that G is a subgroup of some Artin group $\mathcal{A}$. Moreover, using the computer algebra software Magma, we show that G is finitely presented and is isomorphic to a subgroup of infinite index in $\mathcal{A}$. Although we have obtained similar partial results for the triples $(2, 2, 0)$, $(2, 1, 0)$, $(2, 1, 1)$, $(2, 2, 0)$, and $(2, 2, 2)$, we do not include them in this dissertation.

While the three problems discussed above are seemingly disconnected, they are in fact intimately related. They reflect a beautiful interplay between Artin groups and mapping class groups.

CHAPTER 1

MAPPING CLASS GROUPS

1.1 Definition

Throughout this dissertation, we assume that $S = S_{g,b}$ is a connected, compact, orientable surface with genus $0 \leq g < \infty$ and $0 \leq b < \infty$ boundary components. On rare occasions, we will also assume that the surface has p punctures or marked points in its interior. In these cases, $S = S_{g,b,p}$. Unless explicitly stated, all surfaces will be assumed without punctures.

Denote by $Homeo^+(S)$ the group of orientation preserving homeomorphisms of S which are the identity on the boundary ∂S. Let $Homeo_0(S)$ be the subgroup of $Homeo^+(S)$ consisting of all homeomorphisms which are isotopic to the identity, relative to ∂S. Clearly, $Homeo_0(S)$ is a normal subgroup of $Homeo^+(S)$.

Definition 1.1.1. *The* ***mapping class group of S****, denoted by $Mod(S)$, is defined to be the group of isotopy classes of orientation preserving homeomorphisms of S which fix the boundary pointwise. In other words, $Mod(S)$ is the group of orientation preserving homeomorphisms of S which are the identity on ∂S modulo homeomorphisms which are isotopic to the identity by an isotopy which fixes ∂S pointwise. If S has punctures, then we also stipulate that the homeomorphisms preserve the puncture set, and are taken modulo isotopies leaving the puncture set invariant.*

$$\boldsymbol{Mod(S) = Homeo^+(S)/Homeo_0(S)}$$

It should be noted that the mapping class group of S has other equivalent definitions. For example, one may define $Mod(S)$ to be the group of isotopy classes of orientation preserving diffeomorphisms of S which fix the boundary pointwise. Alternatively, one may replace isotopies with homotopies in the definitions above. For more information about this, see theorems 1.9 and 1.10 in [10]. In this dissertation, we will stick to definition 1.1.1 for $Mod(S)$ once and for all.

1.2 Simple closed curves and intersection numbers

Let a be a simple closed curve in an orientable surface S. a is said to be **essential** if it is not nullhomotopic. That is, a cannot be isotoped to a point. Moreover, a is said to be **peripheral** if it is isotopic to a connected component of ∂S.

Let α and β be simple closed curves in S, and denote their isotopy classes by a and b respectively. The **geometric intersection number** of a and b, denoted by $i(a, b)$, is defined to be the minimal number of intersection points between the representatives of a and b. That is,

$$i(a, b) = min|\alpha' \cap \beta'|$$

where $\alpha \sim \alpha'$ (ie α is isotopic to α') and $\beta \sim \beta'$.

Definition 1.2.1. *A collection $\mathcal{C} = \{a_1, \cdots, a_n\}$ of pairwise nonisotopic simple closed curves in S is said to intersect efficiently if no two elements in $\mathcal{C}$ cobound a bigon and no three elements have a common point of intersection. That is, $i(a_j, a_k) = |a_j \cap a_k|$ for all j, k and $a_j \cap a_k \cap a_l = \emptyset$ for distinct j, k, l.*

Let S be an oriented surface. If we orient the curves α and β, another type of intersection can be defined. The **algebraic intersection number** of α and β, denoted by $\hat{i}(\alpha, \beta)$, is defined to be the sum of indices of the intersection points $\alpha \cap \beta$, where an intersection point has index $+1$ if the orientation of α followed by the orientation of β agrees with the orientation of S, and -1 otherwise.

1.3 Dehn twists and their properties

Consider the annulus A in $\mathbb{R}^2$ parametrized by $\{(r,\theta) : 1 \leq r \leq 2\}$. Define the map $T : A \to A$ by

$$T(r,\theta) = (r, \theta + 2\pi r)$$

Clearly, T is a homeomorphism. As a matter of fact, T is a diffeomorphism. Since its Jacobian $\mathcal{J}(r,\theta)$ equals 1, T is orientation preserving. Moreover, it is easy to see that T fixes the boundary ∂A pointwise. See figure 1.1 for the effect of T on a properly embedded arc.

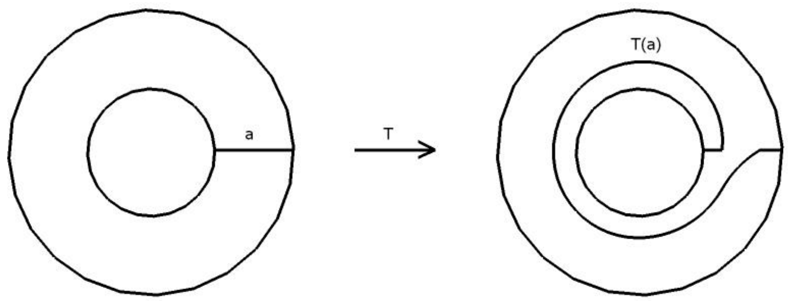


Figure 1.1: The effect of $T : A \to A$ on the arc a.

Definition 1.3.1. *Let α be a simple closed curve in an orientable surface S. Let N be a regular neighborhood of α. If $e : A \to S$ is an orientation preserving embedding with $e(\mathring{A}) = N$ (there are two such embeddings up to isotopy fixing the boundary $\partial\overline{N}$), then the* ***Dehn twist along α*** *is given by the self homeomorphism T_α of S defined as follows:*

$$T_\alpha = \begin{cases} 1 & \text{on } S \setminus N, \\ eTe^{-1} & \text{on } N. \end{cases}$$

It is not hard to check that definition 1.3.1 is independent of the choice of embedding. Also, it should be noted that, in the definition, the homeomorphism T_α depends on the choices of α and N, whereas the isotopy class of T_α is independent of those choices. The

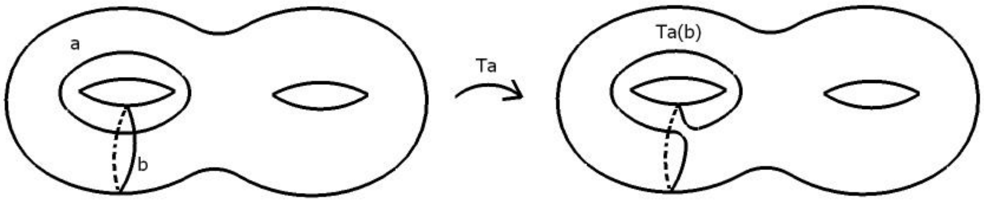


Figure 1.2: The effect of the (left) Dehn twist T_a on the simple closed curve b.

isotopy class of T_α depends only on the isotopy class of α.

We remark that one could define $T : A \to A$ by $T(r, \theta) = (r, \theta - 2\pi r)$. This definition gives rise to a right Dehn twist, as opposed to the left Dehn twist when $T(r, \theta) = (r, \theta + 2\pi r)$.

Intuitively, the effect of a left Dehn twist can be interpreted as follows. If an arc or curve β intersects a simple closed curve α transversally, then the Dehn twist T_α affects β by causing it to turn left (with respect to the orientation of the surface) as it approaches α, turn once around α, then follow β as before.

Notation. Let α be a simple closed curve in S. Throughout this dissertation, we shall abuse notation and refer to the isotopy class $[\alpha]$ simply by α. If $c = [\alpha]$, we will also abuse notation and denote the mapping class T_c by T_α.

Fact 1.3.2. *Suppose that a and b are isotopy classes of simple closed curves in S. Then $T_a = T_b \Leftrightarrow a = b$.*

Fact 1.3.3. *Let $f \in Mod(S)$ and let a be an isotopy class of simple closed curves in S. Then $fT_af^{-1} = T_{f(a)}$.*

Fact 1.3.4. *Suppose $k \in \mathbb{Z}$. If a, b, and c are distinct isotopy classes of simple closed curves in S, then*

$$\left| i(T_a^k(b), c) - |k| i(a,b) i(a,c) \right| \leq i(b,c)$$

Fact 1.3.5. *If $k \in \mathbb{Z}$ and a and b are isotopy classes of simple closed curves in S, then*

$$i(T_a^k(b), b) = |k| i(a,b)^2$$

Proof. Fact 1.3.5 follows immediately from fact 1.3.4 by setting $b = c$. □

Fact 1.3.6. *If T_a is a Dehn twist in $Mod(S)$, then T_a has infinite order.*

Proof. The proof follows from fact 1.3.5 and lemma 7.3.2. Alternatively, fact 1.3.6 is an easy consequence of theorem 7.3.3. □

Proposition 1.3.7. *Suppose that a and b are isotopy classes of simple closed curves in S. Then $T_aT_b = T_bT_a \Leftrightarrow i(a,b) = 0$. The left hand side of this equivalence is called the commutativity or disjointness relation.*

Proof. If $T_aT_b = T_bT_a$, then $T_aT_bT_a^{-1} = T_b$. By fact 1.3.3, this is equivalent to $T_{T_a(b)} = T_b$. By fact 1.3.2, $T_a(b) = b$. But this means that $i(T_a(b), b) = 0$. Moreover, $i(T_a(b), b) = i(a,b)^2$ by fact 1.3.5. Hence $i(a,b)^2 = 0$ and consequently $i(a,b) = 0$.

Conversely, if $i(a,b) = 0$, then the support of T_a may be chosen to be disjoint from b so that $T_a(b) = b$. Thus,

$$T_aT_b = T_aT_bT_a^{-1}T_a = T_{T_a(b)}T_a = T_bT_a$$

where the second equality is due to fact 1.3.3. □

Proposition 1.3.8. *Suppose that a and b are distinct isotopy classes of simple closed curves in S. Then $T_aT_bT_a = T_bT_aT_b \Leftrightarrow i(a,b) = 1$. The left hand side of this equivalence is called the braid relation.*

Proof. The relation $T_aT_bT_a \overset{(1)}{=} T_bT_aT_b$ implies $T_aT_bT_aT_b^{-1}T_a^{-1} = T_b$. By fact 1.3.3, this is equivalent to $T_{T_aT_b(a)} = T_b$. By fact 1.3.2, $T_aT_b(a) = b$, and so $i(a,b) = i(a, T_aT_b(a))$. Applying T_a^{-1} to a and $T_aT_b(a)$, we see that $i(a, T_aT_b(a)) = i(a, T_b(a)) = i(a,b)^2$ where the last equality is due to fact 1.3.5. Hence, $i(a,b)^2 = i(a,b)$ and so $i(a,b) = 0$ or 1. If $i(a,b) = 0$, proposition 1.3.7 implies that $T_aT_b \overset{(2)}{=} T_bT_a$. But (1) and (2) imply $T_a = T_b$. By fact 1.3.2, $a = b$, which contradicts the assumption that a and b are distinct. Therefore, $i(a,b) = 1$.

Conversely, $i(a,b) = 1$ implies that $T_aT_b(a) = b$. One can verify this by drawing pictures. Hence,

$$T_aT_bT_a = T_aT_bT_aT_b^{-1}T_a^{-1}T_aT_b = T_{T_aT_b(a)}T_aT_b = T_bT_aT_a$$

□

Proposition 1.3.9. *Let a and b represent isotopy classes of essential simple closed curves in S, and denote by T_a and T_b their respective Dehn twists in $Mod(S)$. If $T_a^p = T_b^q$ for some $p, q \in \mathbb{Z}$ and $p \neq 0$, then $a = b$.*

Proof. Assume that $a \neq b$. If $i(a,b) > 0$, then

$$0 = i(b,b) = i(T_b^q(b), b) = i(T_a^p(b), b) = |p| i(a,b)^2$$

where the last equality is due to fact 1.3.5. This gives $p = 0$, which is a contradiction. So $i(a,b) = 0$.

If $i(a,b) = 0$ and a is non-peripheral, then lemma 7.3.2 furnishes an isotopy class c such that $i(a,c) > 0$ and $i(b,c) = 0$. Then

$$0 = i(c,c) = i(T_b^q(c), c) = i(T_a^p(c), c) = |p| i(a,c)^2$$

implies that $p = 0$, which is a contradiction.

Finally, suppose that $i(a,b) = 0$ and a is peripheral. Let $\hat{S}$ be the closed surface obtained from S by attaching an $S_{1,1}$ to each connected component of ∂S. The natural homomorphism $i_* : Mod(S) \to Mod(\hat{S})$ which extends by the identity on $\hat{S} \backslash S$ is well-defined. As such, $T_a^p = T_b^q$ in $Mod(\hat{S})$. Moreover, a and b are still essential in $\hat{S}$, and lemma 7.3.1 implies that $a \neq b$ in $\hat{S}$. By lemma 7.3.2, there exists an isotopy class c in $\hat{S}$ such that $i(a,c) > 0$ and $i(b,c) = 0$. As shown above, this implies that $p = 0$, which is a contradiction. So $a = b$ in $\hat{S}$. By lemma 7.3.1, $a = b$ in S. □

Corollary 1.3.10. *If $a \neq b$ and $T_a^p = T_b^q$ for some $p, q \in \mathbb{Z}$, then $p = q = 0$.*

Proof. Proposition 1.3.9 implies $p = 0$, and so $1 = T_b^q$. Now, fact 1.3.6 implies $q = 0$. □

Theorem 1.3.11 (Ishida.)**.** *If $i(a,b) \geq 2$, then there is no relation between T_a and T_b. That is, T_a and T_b generate a free group of rank two.*

Lemma 1.3.12. *If $i(a,b) \geq 2$, then $i(a,c) > i(b,c) \Rightarrow i(a,T_a^n(c)) < i(b,T_a^n(c))$ for all $n \in \mathbb{Z} \setminus \{0\}$.*

Proof. By fact 1.3.4, it follows that

$$\begin{aligned} i(b,T_a^n(c)) &\geq |n| i(a,b) i(a,c) - i(b,c) \\ &> 2i(a,c) - i(a,c) \\ &= i(a,c) \\ &= i(a,T_a^n(c)) \end{aligned}$$

□

Proof of Theorem 1.3.11. If $w \in \langle T_a, T_b \rangle$, then $w = T_b^{m_k} T_a^{n_k} \cdots T_b^{m_1} T_a^{n_1}$, where $m_j, n_j \in \mathbb{Z}$. Assume $w = T_b^{m_k} T_a^{n_k} \cdots T_b^{m_1} T_a^{n_1} = 1$, and note that

1. If $T_a^{n_k} T_b^{m_{k-1}} T_a^{n_{k-1}} \cdots T_b^{m_1} = 1$, then $T_b^{m_{k-1}} T_a^{n_{k-1}} \cdots T_b^{m_1} T_a^{n_k} = 1$.

2. If $T_b^{m_k} T_a^{n_k} T_b^{m_{k-1}} \cdots T_a^{n_2} T_b^{m_1} = 1$, then $T_b^{m_1 + m_k} T_a^{n_k} T_b^{m_{k-1}} \cdots T_a^{n_2} = 1$.

3. If $T_a^{n_k} T_b^{m_{k-1}} T_a^{n_{k-1}} \cdots T_b^{m_1} T_a^{n_1} = 1$, then $T_b^{m_{k-1}} T_a^{n_{k-1}} \cdots T_b^{m_1} T_a^{n_1 + n_k} = 1$.

Hence, up to conjugation, we can assume that all the m_j and all the n_j are nonzero in w. Since $i(a,a) < i(b,a)$, $i(a, T_a^{n_1}(a)) < i(b, T_a^{n_1}(a))$. By repeated applications of lemma 1.3.12, we have:

$$\begin{aligned}
i(a, T_b^{m_1}T_a^{n_1}(a)) &> i(b, T_b^{m_1}T_a^{n_1}(a)) \\
i(b, T_a^{n_2}T_b^{m_1}T_a^{n_1}(a)) &> i(a, T_a^{n_2}T_b^{m_1}T_a^{n_1}(a)) \\
&\vdots \\
i(a, T_b^{m_k}T_a^{n_k}\cdots T_b^{m_1}T_a^{n_1}(a)) &> i(b, T_b^{m_k}T_a^{n_k}\cdots T_b^{m_1}T_a^{n_1}(a))
\end{aligned}$$

The last inequality implies $i(a,a) > i(b,a)$, which is a contradiction. □

We remark that an alternative proof of theorem 1.3.11 can be found in [11].

1.4 Alexander's trick and Dehn twist relations

Alexander's trick states that the mapping class group of the closed disk is trivial. It is a very useful tool which is used in verifying Dehn twist relations in $Mod(S)$ (See, for example, the lantern and chain relations below). Alexander's trick is also used to derive the so called Alexander's method which determines whether two elements f and g in $Homeo^+(S)$ represent the same mapping class in $Mod(S)$. This is done by studying the actions of f and g on a system of simple closed curves and simple (properly embedded) arcs that cut S into a disjoint union of open disks, and then applying Alexander's trick. For more information about the proofs, see [10].

Theorem 1.4.1 (Alexander's Trick.)**.** *The mapping class group of the closed disk is trivial.*

Theorem 1.4.2 (Alexander's Method.)**.** *Suppose S is a compact orientable surface. Let $\{a_1, \cdots, a_n\}$ be a collection of essential, pairwise nonisotopic, oriented simple closed curves and proper arcs in S such that:*

- *$a_1, \cdots, a_n$ fill the surface S. ie. S cut along $\cup a_i$ is a disjoint union of open disks.*
- *a_i and a_j are in minimal position for all $i \neq j$. ie. they do not cobound a bigon.*

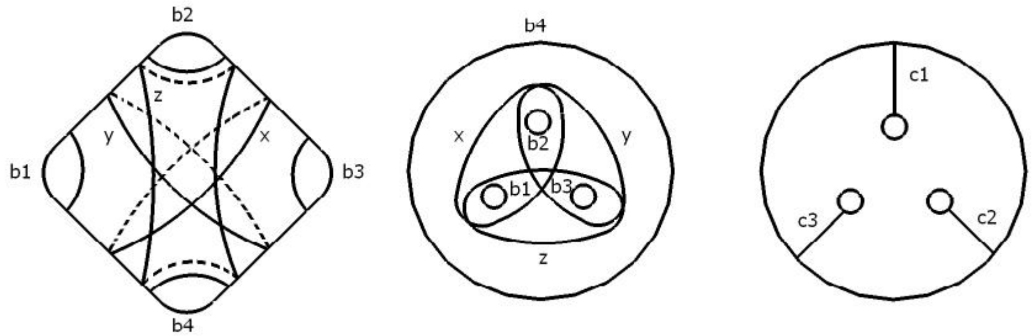


Figure 1.3: According to the lantern relation, $T_xT_yT_z \overset{(*)}{=} T_{b_1}T_{b_2}T_{b_3}T_{b_4}$ in $Mod(S_{0,4})$. $S_{0,4}$ is homeomorphic to $\mathbb{D}_{0,3}$, the closed disk with three boundaries. The middle picture shows x, y, z, and b_i, $i = 1, 2, 3, 4$ in $\mathbb{D}_{0,3}$. Since the arcs c_1, c_2, and c_3 cut $\mathbb{D}_{0,3}$ into a disk, it suffices (by Alexander's trick) to show that $(*)$ holds on these arcs in order to prove theorem 1.4.3.

- *No three members of the collection intersect pairwise. ie. at least one of $a_i \cap a_j$, $a_i \cap a_k$, and $a_j \cap a_k$ is empty when i, j, and k are distinct.*

Let $\phi : S \to S$ be an orientation preserving homeomorphism which fixes ∂S pointwise. Suppose σ is a permutation of $\{1, \cdots, n\}$ such that $\phi(a_i)$ is isotopic to $a_{\sigma(i)}$ relative to ∂S for each i. Then $\phi(\cup a_i)$ is isotopic to $\cup a_i$ relative to ∂S. Call this isotopy F.If we think of $\cup a_i$ as a graph Γ in S whose vertices are intersection points and arc endpoints, then the composition $F \circ \phi$ gives an automorphism $\phi_\star$ of Γ. If $\phi_\star$ fixes each vertex and edge of Γ, with orientations, then ϕ is isotopic to the identity. Otherwise, ϕ is isotopic to a nontrivial finite order homeomorphism.

Theorem 1.4.3 (The Lantern Relation)**.** *Suppose that $S = S_{0,4}$, the sphere with four boundary components, and consider the configuration of simple closed curves shown in Figure 1.3. Then, the following relation holds in $Mod(S)$:*

$$T_xT_yT_z = T_{b_1}T_{b_2}T_{b_3}T_{b_4}$$

It should be noted that this relation is written using functional notation (ie elements on the right are applied first), and the left Dehn twist convention is used.

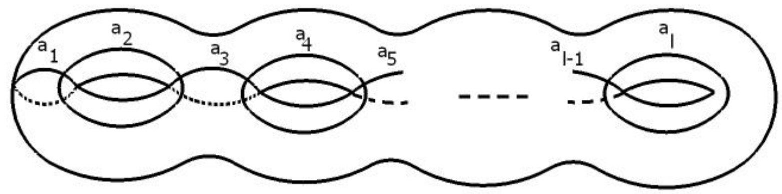


Figure 1.4: The curves $a_1, \cdots, a_n$ form an n-chain.

Sketch of Proof. Since cutting S along the arcs c_1, c_2, and c_3 reduces it to a disk (see figure 1.3), it suffices, by Alexander's trick, to show that the relation holds on $c_i, i = 1, 2, 3$. This can be checked by drawing pictures. □

Definition 1.4.4. *A collection $\{a_1, \cdots, a_l\}$ of pairwise non-isotopic simple closed curves in S forms a chain of length l (l-chain for short) if $i(a_j, a_{j+1}) = 1$ for $j = 1, \cdots, l-1$ and $i(a_j, a_k) = 0$ for $|j-k| \geq 2$.*

Theorem 1.4.5 (The Chain Relation.)**.** *Suppose that a collection $\{a_1, \cdots, a_l\}$ of simple closed curves forms an l-chain in an orientable surface S.*

- *If l is even, then the boundary of a closed regular neighborhood $N_\epsilon(a_1 \cup \cdots \cup a_l)$ consists of one simple closed curve d, and $(T_1 T_2 \cdots T_l)^{2l+2} = T_d$.*
- *If l is odd, then the boundary of a closed regular neighborhood $N_\epsilon(a_1 \cup \cdots \cup a_l)$ consists of two simple closed curves c_1 and c_2, and $(T_1 T_2 \cdots T_l)^{l+1} = T_{c_1} T_{c_2}$.*

Sketch of Proof. Let N_ϵ be a closed regular neighborhood of $\cup_{i=1}^{l} a_i$. The inclusion $i : N_\epsilon \hookrightarrow S$ induces a homomorphism $i_* : Mod(N_\epsilon) \to Mod(S)$ defined by extending by the identity on the compliment (see section 1.5). It follows from the definition of a Dehn a twist and the fact that i_* is a homomorphism that any relation between the T_i in $Mod(N_\epsilon)$ must hold between the T_i in $Mod(S)$. In particular, it suffices to prove the chain relation in $Mod(N_\epsilon)$. To do that, choose a system of arcs and curves that cut N_ϵ into disjoint open disks, then show that the relation holds on this chosen system. By Alexander's method, the theorem follows. □

1.5 Geometric subgroups of Mod(S)

Suppose $S = S_{g,b,p}$ is a connected orientable surface with genus $0 \leq g < \infty$, $0 \leq b < \infty$ boundary components, and $0 \leq p < \infty$ punctures. Let $P = \{x_1, \cdots, x_p\}$ be a set of punctures or marked points in S. Let F be a **subsurface** of S. That is, F is a closed subset of S such that ∂F is contained in the interior of S, and ∂F is disjoint from the set of punctures. A subsurface $F \subset S$ is said to be **essential** if no component of $\overline{S \backslash F}$ is a disk disjoint from P. In other words, F does not split off a disk. The inclusion map $i : (F, F \cap P) \to (S, P)$ induces a natural homomorphism

$$i_* : Mod((F, F \cap P) \to Mod(S, P)$$
$$[h] \mapsto i_*([h])$$

defined by extending by the identity on the compliment. That is, if $[h]$ is the mapping class of the homeomorphism $h : (F, F \cap P) \to (F, F \cap P)$, then $i_*([h])$ is the mapping class represented by extending h to S, by the identity of $S \backslash F$. $Im(i_*)$ is called a **geometric subgroup** of $Mod(S)$.

Definition 1.5.1. *Let $(F, F \cap P)$ be a subsurface of (S, P). A component N of $\overline{S \backslash F}$ is said to be a cylinder exterior to F if N is disjoint from P and both components of ∂N are also components of ∂F (see figure 1.5).*

Theorem 1.5.2 (Rolfsen-Paris)**.** *Suppose that $F \subset S$ is an essential subsurface, and consider the natural homomorphism*

$$i_* : Mod((F, F \cap P) \to Mod(S, P)$$

- *If $(F, F \cap P)$ is a disk with $|F \cap P| \leq 1$, then i_* is injective by theorem 1.4.1.*
- *If $(F, F \cap P)$ is an annulus disjoint from P and $(F, F \cap P)$ splits off a disk with one puncture, then $ker(i_*) = Mod(F, F \cap P)$. Otherwise, i_* is injective. (Note that $(F, F \cap P)$ cannot split off a disk because it is essential in (S, P)).*

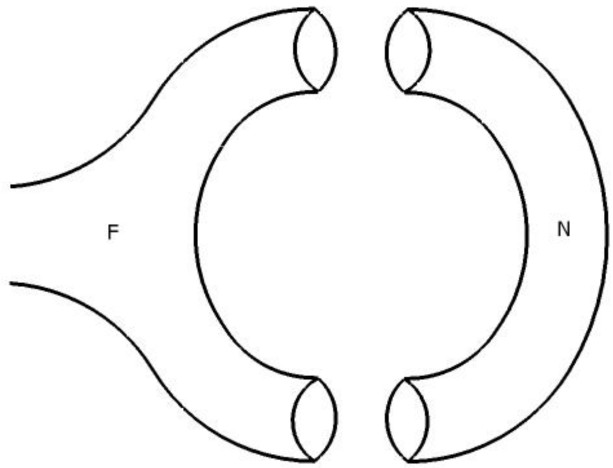


Figure 1.5: N is a cylinder exterior to F.

- *If $(F, F \cap P)$ is not as in the above two cases, let $a_1, \cdots, a_r$ denote the boundary components of $(F, F \cap P)$ which split off once punctured disks, and let $b_j, b'_j, j = 1, \cdots, s$ be the boundary component pairs which cobound exterior cylinders A_j. Then $ker(i_*)$ is generated by $\{T_{a_1}, \cdots, T_{a_r}, T_{b_1}T_{b'_1}^{-1}, \cdots, T_{b_s}T_{b'_s}^{-1}\}$, and is isomorphic to $\mathbb{Z}^{r+s}$.*

Corollary 1.5.3. *Let $F \subset S$ be any subsurface and consider the natural homomorphism*

$$i_* : Mod((F, F \cap P) \to Mod(S, P)$$

- *If $(F, F \cap P)$ is a disk with one or no punctures, then i_* is injective.*

- *If $(F, F \cap P)$ is an annulus disjoint from P, then i_* is injective iff no component of $\overline{S \backslash F}$ is a disk with less than two punctures.*

- *If $(F, F \cap P)$ is neither of the subsurfaces above, then i_* is injective iff no component of $\overline{S \backslash F}$ is a cylinder exterior to F or a disk with less than two punctures.*

1.6 The center of Mod(S)

In this section, we state a theorem about the center of the mapping class group of an orientable surface. For more information, we refer the reader to [23] and [10]. Recall that the center of a group G is the subgroup given by $Z(G) = \{x \in G \mid gx = xg \; \forall \; g \in G\}$.

Theorem 1.6.1. *Let $S = S_{g,b,p}$ be an orientable surface of genus g with b boundary components and p punctures.*

(1) *If $S = S_{0,0,p}$ with $p = 0, 1$, then $ZMod(S) = Mod(S) = \{1\}$.*

(2) *If $S = S_{0,0,2}$, then $ZMod(S) = Mod(S) \cong \mathbb{Z}/2\mathbb{Z}$.*

(3) *If $S = S_{0,0,3}$, then $Mod(S) \cong \Sigma_3$. Hence, $ZMod(S) = \{1\}$.*

(4) *If $S = S_{0,1,p}$ with $p = 0, 1$, then $ZMod(S) = Mod(S) = \{1\}$.*

(5) *If $S = S_{0,1,2}$, then $ZMod(S) = Mod(S) \cong \mathbb{Z}$.*

(6) *If $S = S_{0,2,0}$, then $ZMod(S) = Mod(S) \cong \mathbb{Z}$.*

(7) *If $S = S_{0,3,0}$, then $ZMod(S) = Mod(S) \cong \mathbb{Z}^3$.*

(8) *If $S = S_{1,0,0}$, then $Mod(S) \cong SL_2(\mathbb{Z})$. Hence, $ZMod(S) \cong \mathbb{Z}/2\mathbb{Z}$.*

(9) *If $S = S_{1,0,1}$, then $Mod(S) \cong SL_2(\mathbb{Z})$. Hence, $ZMod(S) \cong \mathbb{Z}/2\mathbb{Z}$.*

(10) *If $S = S_{1,0,2}$, then $ZMod(S) \cong \mathbb{Z}/2\mathbb{Z}$.*

(11) *If $S = S_{1,1,0}$, then $ZMod(S) \cong \mathbb{Z}$.*

(12) *If $S = S_{2,0,0}$, then $ZMod(S) \cong \mathbb{Z}/2\mathbb{Z}$.*

Now suppose that S is different from the surfaces in (1) − (12). That is, S is not of the following: The sphere with less than four punctures, the disk with less than three punctures, the annulus with no punctures, the torus with less than three punctures, the torus with one boundary component and no punctures, and the closed genus two surface with no punctures. Denote by $a_1, \cdots, a_b$ the isotopy classes of all the peripheral curves in S and by T_i the Dehn twist along a_i, $i = 1, \cdots, b$. Then the center $ZMod(S)$ of $Mod(S)$ is the subgroup generated by $T_1, \cdots, T_b$ and is isomorphic to the free abelian group $\mathbb{Z}^b$. In particular, if $S = S_{g,0,p}$ with $g \geq 3$ and $p \geq 0$, then $ZMod(S)$ is trivial.

CHAPTER 2

BRAID GROUPS

2.1 The classical braid groups $\mathcal{B}_n$

We think of a classical braid β on n strands (n-braid for short) as a continuous family of n disjoint embedded paths $f_i : I \to \mathbb{R}^2 \times I$ (called strands) in $\mathbb{R}^3$, starting at the points $\{(j,0,1)\}_{j=1}^{n+1}$ and ending at $\{(\sigma(j),0,0)\}_{j=1}^{n+1}$, where $\sigma \in \Sigma_n$. Each strand is a monotonically decreasing function in the coordinate z. So the paths of β run monotonically down the z-axis, while possibly twisting around each other. This definition gives rise to the so called geometric or physical braid (See Figure 2.1).

Two n-braids β and β' are said to be equivalent, if one can be deformed to the other by a braid isotopy. In other words, there is a continuous family $F_t : \mathbb{R}^2 \times I \to \mathbb{R}^2 \times I$ which is the identity on $\mathbb{R}^2 \times \{0\}$ and $\mathbb{R}^2 \times \{1\}$ for all $i \in I$ (ie F_t fixes the endpoints), $F_t(\beta)$ is an n-braid for all $t \in I$, and $F_0(\beta) = \beta$ and $F_1(\beta) = \beta'$.The set of isotopy classes of n-braids forms a group under braid concatenation or stacking. Given two n-braids β and β', we stack β on top of β' and rescale t to obtain the product braid $\beta.\beta' : I \to \mathbb{R}^2 \times I$. The classical braid group of n strands is denoted by $\mathcal{B}_n$.

Consider a n-braid β in $\mathbb{R}^3$. Project β onto the xz-plane. This projection may be performed so that, as one moves down the z-axis, only one crossing is encountered at a time. The resulting two dimensional picture of this projection is called a braid diagram for β. In a braid diagram, what appears to be a disconnected strand at a crossing is meant to

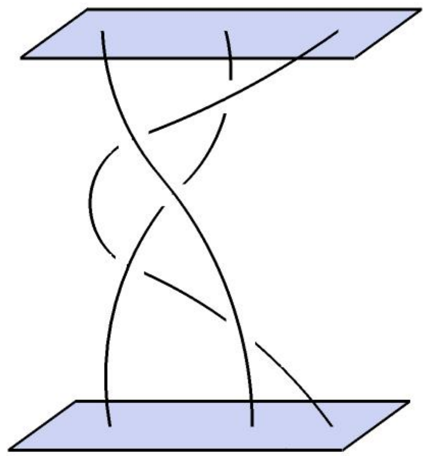


Figure 2.1: A geometric braid on 3 strands.

be a strand which comes from behind the connected strand (See Figure 2.2).

The group $\mathcal{B}_n$ is generated by $n-1$ braids $\gamma_1, \cdots, \gamma_{n-1}$, where γ_i represents the i^{th} strand crossed over the $(i+1)^{st}$. The generators γ_i satisfy two types of relations, namely $\gamma_i \gamma_j = \gamma_j \gamma_i$ whenever $|i-j| \geq 2$ and $\gamma_i \gamma_{i+1} \gamma_i = \gamma_{i+1} \gamma_i \gamma_{i+1}$ for $i = 1, \cdots, n-2$. These relations are illustrated in Figure 2.2. This gives a presentation for $\mathcal{B}_n$:

$$\mathcal{B}_n = \langle \gamma_1, \cdots, \gamma_n \mid \gamma_i \gamma_{i+1} \gamma_i = \gamma_{i+1} \gamma_i \gamma_{i+1}, \gamma_i \gamma_j = \gamma_j \gamma_i \ if \ |i-j| \geq 2 \rangle$$

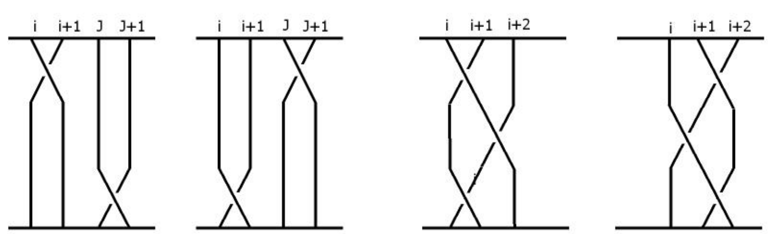


Figure 2.2: $\gamma_i \gamma_j = \gamma_j \gamma_i$ when $|i-j| \geq 2$ (left) and $\gamma_i \gamma_{i+1} \gamma_i = \gamma_{i+1} \gamma_i \gamma_{i+1}$ (right).

We end this section with a fact that shall be used in sections 5.5 and 7.8. This fact states that $\mathcal{B}_{n+1}$ is isomorphic to the finite type Artin group $\mathcal{A}(A_n)$, defined in chapter 3. The isomorphism is given explicitly in theorem 5.5.1.

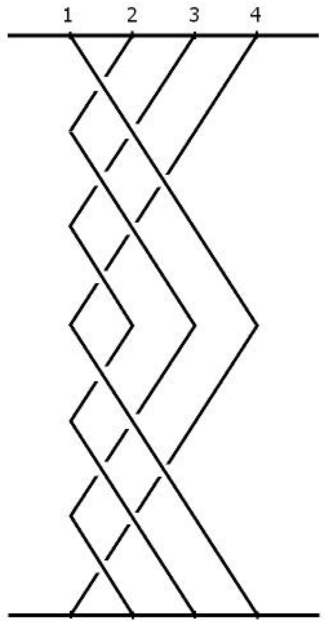


Figure 2.3: The geometric 4-braid $(\gamma_1\gamma_2\gamma_3)^4$, which generates $Z(\mathcal{B}_4)$.

2.2 The center of $\mathcal{B}_n$

We state a theorem about the center of the braid group $\mathcal{B}_n$ on n strands. For the proof, we refer the reader to [18] or [4].

Theorem 2.2.1. *If $n \geq 3$, then the center $Z(\mathcal{B}_n)$ of $\mathcal{B}_n$ is infinite cyclic generated by $x = (\gamma_1\gamma_2\cdots\gamma_{n-1})^n$.*

Geometrically, the generator of $Z(\mathcal{B}_n)$ can be obtained from the trivial n-braid by fixing the top and rotating the bottom by 2π. Figure 2.3 illustrates $(\gamma_1\gamma_2\gamma_3)^4$, the generator of $Z(\mathcal{B}_4)$.

To see why x commutes with every element in $\mathcal{B}_n$, pick an arbitrary n-braid y and consider xyx^{-1}. By twisting y (in xyx^{-1}) by -2π, one can check that the resulting n-braid is isotopic to y.

2.3 Annular braid groups

Annular braids are defined in analogy with classical braids. An annular n-braid is a continuous collection of n disjoint embedded paths in $\mathbb{R}^3$, starting at equally spaced points on the unit circle in the plane $z = 1$ and ending at analogous points on the unit circle in the xy-plane. The paths run down monotonically with respect to the z-axis, while possibly twisting around one another. However, we require that an annular braid never intersect the z-axis. Thus, annular braids live in $\mathbb{R}^3$ minus the z-axis.

Two annular n-braids α and α' are said to be equivalent, if we can deform one to the other by a braid isotopy. That is, there exists a continuous family $G_t : \mathbb{R}^2 \setminus \{0\} \times I \to \mathbb{R}^2 \setminus \{0\} \times I$ such that, for all $t \in I$, G_t fixes the endpoints and $\alpha_t = G_t(\alpha)$ is an annular n-braid. Moreover, $\alpha_0 = \alpha$ and $\alpha_1 = \alpha'$. The set of isotopy classes of annular n-braids forms a group under the stacking operation. This group is denoted by CB_n. It was shown by Crisp [7] that CB_n is isomorphic to the finite type Artin group $\mathcal{A}(B_n)$, defined in chapter 3.

For an annular braid, rather than projecting onto a plane, we thicken the z-axis to form a cylinder, project the strands onto the cylinder's surface, and view the projection from outside. The group CB_n has generators $\sigma_0, \sigma_1, \sigma_2, \cdots, \sigma_{n-1}$, and τ, where σ_i, $i = 0, \cdots, n-1$ represents crossing the i^{th} strand over the $(i+1)^{st}$ modulo n. In particular, σ_0 represents crossing the n^{th} (or zeroth modulo n) strand over the first. The generator τ is shown in Figure 2.4.

Theorem 2.3.1 (Kent-Peifer). *The annular braid group on n strands, CB_n has presentation:*

$$\begin{aligned} \mathcal{P} \;=\; & \langle \sigma_0, \cdots, \sigma_{n-1}, \tau |\; \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \;\; mod(n) \; for \;\; i = 0, 1, \cdots, n-1 \\ & \sigma_i\sigma_j = \sigma_j\sigma_i \; for \; |i-j| \neq 1, n-1, \;\; \tau\sigma_i\tau^{-1} = \sigma_{(i+1)mod(n)} \rangle \end{aligned}$$

In the presentation $\mathcal{P}$, set $H = \langle \tau \rangle$ and denote by N the normal subgroup of CB_n generated by $\{\sigma_0, \sigma_1, \cdots, \sigma_{n-1}\}$. It is clear from the presentation $\mathcal{P}$ that conjugating

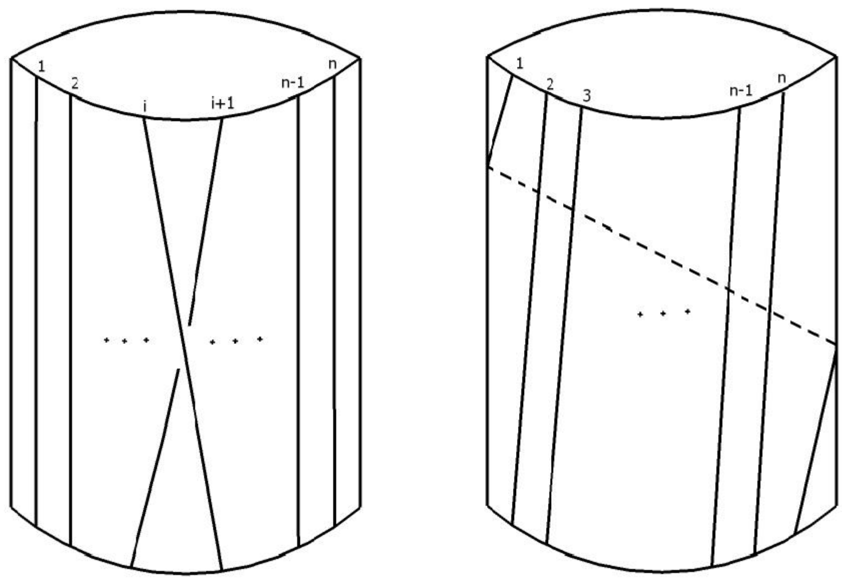


Figure 2.4: This picture illustrates the generators σ_i (left) and τ (right) of the annular braid group CB_n. In CB_n, the defining relations are $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ modulo n, $i = 0, \cdots, n-1$, $[\sigma_i, \sigma_j] = 1$ for $|i-j| \neq 1, n-1$, and $\tau\sigma_i\tau^{-1} = \sigma_{i+1}$ modulo n.

$\{\sigma_0, \sigma_1, \cdots, \sigma_{n-1}\}$ by τ induces an automorphism of N. This gives a homomorphism $\phi : H \to Aut(N)$, where $\phi(\tau)(\sigma_i) = \tau\sigma_i\tau^{-1} = \sigma_{(i+1)mod(n)}$, $i = 0, \cdots, n-1$. As such, CB_n is isomorphic to the semidirect product $N \rtimes_\phi H$. It follows from the structure of CB_n and its presentation $\mathcal{P}$ that N is isomorphic to the affine Artin group $\mathcal{A}(\widetilde{A}_{n-1})$ (see chapter 3). Therefore, $CB_n \cong \mathcal{A}(\widetilde{A}_{n-1}) \rtimes \langle\tau\rangle$.

Consider $\mathcal{B}_{n+1}$, the classical braid group on $n+1$ strands. Let D_{n+1} be the set of all classical $(n+1)$-braids, where the endpoint of the first strand does not get permuted. In other words, the first strand begins and ends at the first position. D_{n+1} is a finite index subgroup of $\mathcal{B}_{n+1}$, which was studied by Chow [6]. To see why $[\mathcal{B}_{n+1} : D_{n+1}] < \infty$, recall that the pure braid group $\mathcal{PB}_{n+1}$ consists of all (n+1)-braids where the i^{th} strand ends up at the i^{th} position for all $i = 1, \cdots, n+1$. Algebraically, $\mathcal{PB}_{n+1}$ is the kernel of the epimorphism $\mathcal{B}_{n+1} \to \Sigma_{n+1}$ given by $\gamma_i \mapsto (i\; i+1)$. Thus, $\mathcal{PB}_{n+1}$ is a subgroup of index $(n+1)!$ in $\mathcal{B}_{n+1}$. Since $\mathcal{PB}_{n+1}$ is a subgroup of D_{n+1}, D_{n+1} has finite index in $\mathcal{B}_{n+1}$. Chow's

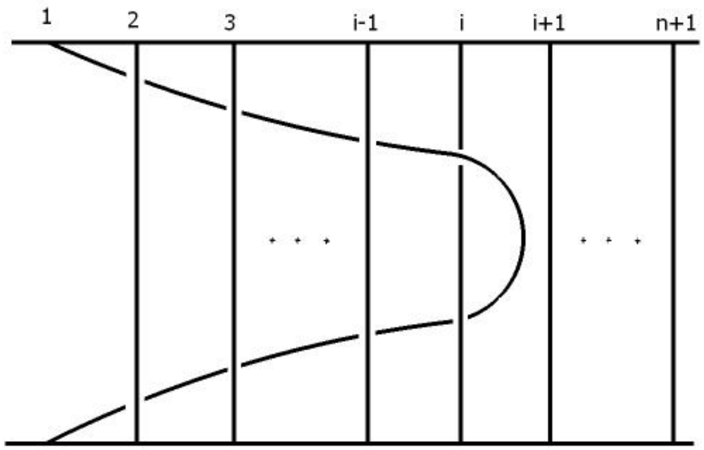


Figure 2.5: The generator a_i of D_{n+1}. As seen in the picture, the braid a_i starts at the first position, goes behind strands 2 through $i-1$, crosses over then under strand i, and then goes back to the first position from behind strands $i-1$ through 2.

presentation of D_{n+1} is:

$$\begin{aligned} \mathcal{D} \;\; = \;\; & \langle \gamma_2, \gamma_3, \cdots, \gamma_n, a_2, a_3, \cdots, a_{n+1} | \; \gamma_i\gamma_{i+1}\gamma_i = \gamma_{i+1}\gamma_i\gamma_{i+1} \;, \\ & \gamma_i\gamma_j = \gamma_j\gamma_i \; for \; |i-j| \geq 2, \;\; \gamma_i a_k \gamma_i^{-1} = a_k \;\; for \;\; k \neq i, i+1 \\ & \gamma_i a_i \gamma_i^{-1} = a_{i+1}, \;\; \gamma_i a_{i+1} \gamma_i^{-1} = a_{i+1}^{-1} a_i a_{i+1} \rangle \end{aligned}$$

In the presentation $\mathcal{D}$, γ_i represents the standard generator of $\mathcal{B}_{n+1}$, where the i^{th} strand is crossed over the $(i+1)^{st}$. Note, however, that the index i begins at 2, and thus none of the crossings include the first strand. The generators a_i are the ones involving the first strand. More precisely, a_i is the braid corresponding to the first strand going behind the 2^{nd} through the $(i-1)^{st}$ strands, crossing over then under the i^{th} strand, then returning back to the first position from behind the $(i-1)^{st}$ strand through the 2^{nd}. See Figure 2.5 for illustration.

Theorem 2.3.2 (Kent-Peifer)**.** *The groups CB_n and D_{n+1} are isomorphic.*

Proof. Let $y \in D_{n+1}$. Then y is a classical $(n+1)$-braid with the stipulation that endpoint of the first strand does not get permuted. Thicken the first strand of y to form a cylinder, then pull it tight. Now wrap the remaining strands of y once around the cylinder's surface. After the wrapping occurs, the positions of strands 2 through $n+1$ reverse order. In other words, strand 2 becomes strand $n-1$ on the cylinder, strand 3 becomes strand $n-2$, $\cdots$, and strand $n+1$ becomes strand 0 (See Figure 2.6). While wrapping, strands are neither allowed to pass through one another nor through the cylinder. The result of this process is an annular n-braid. Define $\Phi : D_{n+1} \to CB_n$ by

$$\gamma_i \mapsto \sigma_{n-i} \text{ for } i = 2, \cdots, n$$

$$a_2 \mapsto (\sigma_{n-2}\sigma_{n-3}\cdots\sigma_0\tau)^{-1}$$

$$a_j \mapsto (\sigma_{(n+1)-j}\cdots\sigma_{n-2})(\sigma_{n-2}\sigma_{n-3}\cdots\sigma_0\tau)^{-1}(\sigma_{(n+1)-j}\cdots\sigma_{n-2})^{-1}$$

$$\text{for } j = 3, \cdots, n+1$$

Let α_1 and α_2 be elements of D_{n+1}. Wrapping $\alpha_1.\alpha_2$ around its thickened first strand yields the same annular braid as when α_1 and α_2 are each wrapped around their thickened first strands, then concatenated in CB_n. Thus, Φ is a homomorphism. It can be checked that τ is the image of $(a_2\gamma_2\cdots\gamma_n)^{-1}$ and σ_{n-1} is the image of $(a_2\gamma_2\cdots\gamma_n)\gamma_n(a_2\gamma_2\cdots\gamma_n)^{-1}$. So Φ is surjective. If $\alpha \in ker(\Phi)$, then $\Phi(\alpha)$ can be deformed into the trivial annular n-braid. Performing an analogous deformation on α yields the trivial $(n+1)$-braid. That is, suppose, for example, $\Phi(\alpha)$ is deformed to the trivial annular braid by pulling its 2^{nd} strand from behind the 3^{rd} strand (thus eliminating two crossings). Then the analogous deformation on α would be pulling the $(n-1)^{st}$ strand from above the $(n-2)^{nd}$. Therefore, Φ is injective, and consequently an isomorphism. □

Theorem 2.3.3 (Kent-Peifer)**.** *The presentation $\mathcal{D}$ of D_{n+1} is equivalent to $\mathcal{P}$*

We refer the reader to [19] for the proof of theorem 2.3.3. It is easy to see that theorem 2.3.1 is an immediate consequence of theorems 2.3.2 and 2.3.3.

In light of the results in this section, we emphasize the following observation:

$$\mathcal{A}(\widetilde{A}_{n-1}) \cong N < CB_n \cong D_{n+1} < \mathcal{B}_{n+1}$$

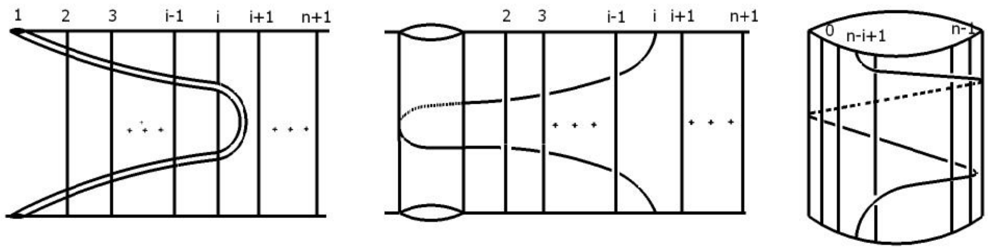


Figure 2.6: Mapping $a_i \in D_{n+1}$ to $\Phi(a_i) \in CB_n$.

where N is the normal subgroup of CB_n generated by $\sigma_0, \cdots, \sigma_{n-1}$ (see theorem 2.3.1) and $\mathcal{A}(\widetilde{A}_{n-1})$ is the affine Artin group of type $\widetilde{A}_{n-1}$ defined in chapter 3.

In section 5.5, we will use this observation along with theorem 5.1.2 to give and embedding of $\mathcal{A}(\widetilde{A}_{n-1})$ into $Mod(S)$.

CHAPTER 3

ARTIN GROUPS

3.1 Artin monoids and Artin groups

A Coxeter system of rank n is a pair (W, S) consisting of a finite set S of order n and a group W with presentation

$$\langle S \mid (st)^{m_{st}} = 1, \;\; for \;\; s, t \in S \;\; such \;\; that \;\; m_{st} \neq \infty \rangle$$

where $m_{ss} = 1$ and $m_{st} = m_{ts} \in \{2, 3, \cdots, \infty\}$ for $s \neq t$. $m_{st} = \infty$ means that there is no relation between s and t. An equivalent presentation is given by:

$$\langle S \mid s^2 = 1 \; \forall \; s \in S, \;\; prod(s, t; m_{st}) = prod(t, s; m_{st}) \;\; such \;\; that \;\; m_{st} \neq \infty \rangle$$

where, again, $m_{ss} = 1$, $m_{st} = m_{ts} \in \{2, 3, \cdots, \infty\}$ for $s \neq t$, and $prod(s, t; m_{st}) = sts \cdots$ where the product contains m_{st} terms.

A Coxeter system is determined by its Coxeter graph Γ. This graph has vertex set S and includes an edge labeled m_{st}, between s and t, whenever $m_{st} \geq 3$. The label $m_{st} = 3$ is usually omitted. The graph Γ defines the type of a Coxeter group. We say that W is a Coxeter group of type Γ, and denote it by $W(\Gamma)$. Alternatively, a Coxeter system can be uniquely determined by its Coxeter matrix $M = (m_{ij})_{i,j \in S}$, where M is an $n \times n$ symmetric matrix with ones on the main diagonal and entries in $\{2, \cdots, \infty\}$ elsewhere. When W is finite, we refer to it as a Coxeter group of finite type. Otherwise, W is of infinite type.

The Artin group, $\mathcal{A}(\Gamma)$, of type Γ has presentation

$$\langle S \mid prod(s,t;m_{st}) = prod(t,s;m_{st}) \;\; such \;\; that \;\; m_{st} \neq \infty \rangle \; (*)$$

It is clear from the presentations that $W(\Gamma)$ is a quotient of $\mathcal{A}(\Gamma)$. It is the quotient of $\mathcal{A}(\Gamma)$ by the normal closure of $\{s^2 | s \in S\}$. We say that an Artin group has finite type, if its corresponding Coxeter group is finite.

Consider $F(S)^+$, the free monoid (semigroup with 1) of positive words in the alphabet of S. The Artin monoid $\mathcal{A}^+(\Gamma)$ of type Γ is obtained from $F(S)^+$ by stipulating that $prod(s,t;m_{st}) \doteq prod(s,t;m_{ts})$ for all $s,t \in S$ and $m_{st} \neq \infty$. The equality $\doteq$ denotes the positive word equivalence in $\mathcal{A}^+(\Gamma)$ (as opposed to the word equivalence in the group $\mathcal{A}(\Gamma)$ which is denoted by $=$). In other words, $\mathcal{A}^+(\Gamma)$ is given by $(*)$, considered as a monoid presentation.

Definition 3.1.1. *Let M and N be monoids. A map $\phi : M \to N$ is said to be a monoid homomorphism if $f(xy) = f(x)f(y)$ for all $x, y \in M$ and $f(1_M) = 1_N$.*

We state some useful definitions and results about Artin monoids and Artin groups. The interested reader is referred to [4] for detailed information.

- If Γ is of finite type, then the canonical homomorphism $\mathcal{A}^+(\Gamma) \to \mathcal{A}(\Gamma)$ is injective.
- The Artin monoid $\mathcal{A}^+(\Gamma)$ is **cancellative**. That is, $UA_1V \doteq UA_2V$ implies $A_1 \doteq A_2$.
- Let $U, V \in \mathcal{A}^+(\Gamma)$. We say that U **divides** V (on the left), and write $U|V$, if $V \doteq UV'$ for some $V' \in \mathcal{A}^+(\Gamma)$.
- An element V is said to be a **common multiple** for a finite subset $\mathcal{U} = \{U_1, \cdots, U_r\}$ of $\mathcal{A}^+(\Gamma)$ if $U_i|V$ for each $i = 1, \cdots, r$. It is shown in [4] that if a common multiple of $\mathcal{U}$ exists, then there exists a necessarily unique least common multiple of $\mathcal{U}$. The least common multiple is a common multiple which divides all the common multiples of $\mathcal{U}$, and is denoted by $[U_1, \cdots, U_r]$. For each pair of elements $s, t \in S$ with $m_{st} \neq \infty$, the least common multiple $[s,t] \doteq prod(s,t;m_{st})$. If $m_{st} = \infty$, then s and t have no common multiple.

- Every element w of $F(S)^+$ can be expressed uniquely as a word in the alphabet of S. The number of letters in this word is called the length of w and is denoted by $l(w)$. Define $l(1) = 0$. It is obvious that $l(s) = 1$ for all $s \in S$ and that $l(ww') = l(w) + l(w')$ for all $w, w' \in F(S)^+$. Define $l : \mathcal{A}^+(\Gamma) \to \mathbb{N}$ by $l(U)$ equals the length of any word in $F(S)^+$ representing U. The function l is well-defined because applying $prod(s,t;m_{st}) = prod(t,s;m_{st})$ to a word $u \in \mathcal{A}^+(\Gamma)$ does not alter its length. Clearly, $U|V$ implies $l(U) \le l(V)$.

Lemma 3.1.2 (Brieskorn-Saito)**.** *Let (W,S) be a Coxeter system with Coxeter graph Γ. If $X, Y \in \mathcal{A}^+(\Gamma)$ and $s,t \in S$ satisfy $sX \doteq tY$, then $\exists\, W \in \mathcal{A}^+(\Gamma)$ such that $X \doteq prod(t,s;m_{st}-1)W$ and $Y \doteq prod(s,t;m_{st}-1)W$.*

Lemma 3.1.2 is called the **Reduction Lemma**. It will be used repeatedly in lemma 5.2.1.

We end this section with a list of Coxeter graphs that are relevant to this dissertation. In what follows, we shall encounter the Coxeter graphs A_n, B_n, D_n, H_3, $I_2(k)$, and $\widetilde{A}_{n-1}$ shown below. All of those graphs are of finite type, except for $\widetilde{A}_{n-1}$. See [14] for a complete classification of Coxeter groups.

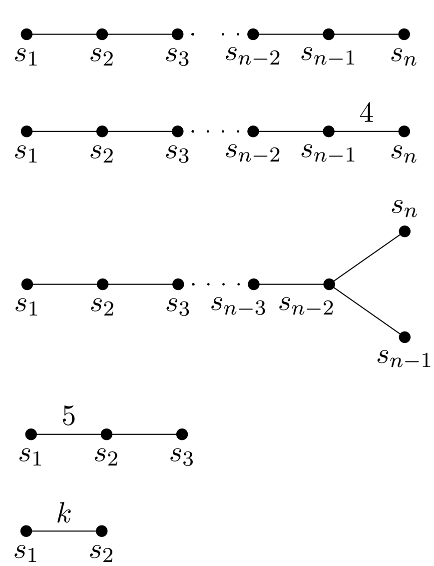
$A_n =$ $(n \ge 2)$

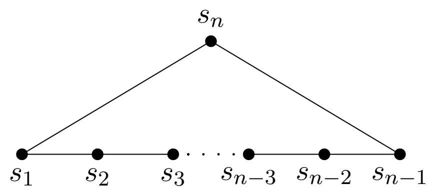
$B_n =$ $(n \ge 3)$

$D_n =$ $(n \ge 4)$

$H_3 =$

$I_2(k) =$ $(k \ge 3)$



$\widetilde{A}_{n-1} =$  $(n \geq 3)$

3.2 LCM-Homomorphisms and foldings

The majority of definitions and results from this section are due to Crisp. The reader is referred to [7] for more information.

Definition 3.2.1. *An Artin monoid homomorphism* $\phi : \mathcal{A}^+(\Gamma) \to \mathcal{A}^+(\Gamma')$ *respects lcms if*

1. $\phi(s) \neq 1$ *for each generator* s*, and*

2. *For each pair of generators* $s, t \in S$*, the pair* $\phi(s), \phi(t)$ *have a common multiple only if* s *and* t *do. In that case,* $[\phi(s), \phi(t)] \mathrel{\underset{\cdot}{=}} \phi([s,t])$.

Theorem 3.2.2 (Crisp)**.** *A homomorphism* $\phi : \mathcal{A}^+(\Gamma) \to \mathcal{A}^+(\Gamma')$ *between Artin monoids which respects lcms is injective.*

Theorem 3.2.3 (Crisp)**.** *If* $\phi : \mathcal{A}^+(\Gamma) \to \mathcal{A}^+(\Gamma')$ *is a monomorphism between finite type Artin monoids, then the induced homomorphism* $\phi_A : \mathcal{A}(\Gamma) \to \mathcal{A}(\Gamma')$ *between Artin groups is injective.*

Let $\mathcal{A}^+(\Gamma)$ be an Artin monoid with generating set S. If $T \subseteq S$ has a common multiple, then T has a unique least common multiple. Denote this least common multiple by Δ_T. Δ_T is also called the fundamental element for T. It was shown in [4] that Δ_T exists if and only if the parabolic subgroup W_T (ie the subgroup of $W(\Gamma)$ generated by T) is finite. When it exists, Δ_T corresponds to the longest element of W_T.

Definition 3.2.4. *Let* (W, S) *and* (W', S') *be Coxeter systems of types* Γ *and* Γ' *respectively, and assume* $m_{st} \neq \infty$ *for all* $s, t \in S$*. Let* $\{T(s) | s \in S\}$ *be a collection of mutually disjoint subsets of* S' *such that*

1. *for each* $s \in S$, $T(s)$ *is nonempty and* $\Delta_{T(s)}$ *exists, and*

2. $prod(\Delta_{T(s)}, \Delta_{T(t)}; m_{st}) \doteq prod(\Delta_{T(t)}, \Delta_{T(s)}; m_{ts}) \doteq [\Delta_{T(s)}, \Delta_{T(t)}]$ *for all* $s, t \in S$.

Define a homomorphism $\phi_T : \mathcal{A}^+(\Gamma) \to \mathcal{A}^+(\Gamma')$ *by* $\phi_T(s) \doteq \Delta_{T(s)}$ *for* $s \in S$. *Such a homomorphism is called an LCM-homomorphism.*

It is clear from condition 2 that ϕ_T is a homomorphism (For each relation R in $\mathcal{A}^+(\Gamma)$, $\phi_T(R)$ is a relation in $\mathcal{A}^+(\Gamma')$). Additionally, ϕ_T respects lcms. Indeed, condition 1 of definition 3.2.1 is satisfied because $T(s) \neq \emptyset$ consists of generators of S'. As such $\Delta_{T(s)} \neq 1$. Moreover, the assumption $m_{st} \neq \infty$ for all s, t guarantees the existence of $[s, t]$ for all s, t. Also, condition 2 of definition 3.2.4 implies the second condition of definition 3.2.1. Since LCM-homomorphisms respect lcms, they are injective by theorem 3.2.2.

Let (W, S) be an irreducible Coxeter system (ie its Coxeter graph Γ is connected), with $S = \{s_1, s_2, \cdots, s_n\}$. A **Coxeter element** h of W is defined to be a product $s_{\sigma(1)}s_{\sigma(2)} \cdots s_{\sigma(n)}$, where $\sigma \in \Sigma_n$. It is known that all Coxeter elements are conjugate in W (see p.74 in [14]). Hence, all the Coxeter elements have the same order in W. Consequently, the **Coxeter number** of W is defined to be the order of a Coxeter element. It is well known [14] that the Coxeter graphs A_n, B_n, D_n, and $I_2(n)$ have Coxeter numbers $n+1$, $2n$, $2n-2$, and n respectively.

Definition 3.2.5. *Let* $\epsilon = I_2(m)$ *with* $m \geq 3$ *and let* k *be a positive integer. Denote by* $k.\epsilon$ *the disjoint union of* k *copies of* ϵ. *The map* $f_\epsilon : k.\epsilon \to \epsilon$ *which sends each copy of* ϵ *in* $k.\epsilon$ *identically to* ϵ *is called a* ***k-fold trivial folding***.

Definition 3.2.6. *Let* $\epsilon = I_2(m)$ *with* $m > 3$ *and let* K *be an irreducible finite type Coxeter graph with Coxeter number* m. *Choose a partition* $K_s \cup K_t$ *of the vertex set of* K *so that there are no edges between the vertices of* K_s *and no edges between the vertices of* K_t. *The* ***dihedral folding*** *of* K *onto* ϵ *is the unique simplicial map* $f_\epsilon : K \to \epsilon$ *such that* $f_\epsilon(K_s) = s$ *and* $f_\epsilon(K_t) = t$.

It is well known [14] that all finite type Coxeter graphs are bipartite. As such, one may always choose a partition as in definition 3.2.6. This partition is unique up to relabeling of the two sets K_s and K_t.

Definition 3.2.7. *Let Γ and Γ' be Coxeter graphs with respective vertex sets S and S'. A **folding** of Γ' onto Γ is a surjective simplicial map $f : \Gamma' \to \Gamma$ such that for every edge $\epsilon = I_2(m)$ with $m \geq 3$, the restriction f_ϵ of f to $f^{-1}(\epsilon)$ is either a k-fold trivial folding or a dihedral folding.*

Remark. Let $f : \Gamma' \to \Gamma$ be a folding so that, for some edge $\epsilon = I_2(m)$ with $m > 3$ in Γ, the restriction f_ϵ of f to $f^{-1}(\epsilon)$ is a dihedral folding and f is trivial otherwise. Since f_ϵ depends on the choice of labeling the partition of $f^{-1}(\epsilon) = K$ into K_s and K_t as in definition 3.2.6, this could possibly give rise to two distinct foldings of Γ' onto Γ. When distinct, we distinguish these foldings by writing $(K, +\epsilon)$ and $(K, -\epsilon)$. See section 3.4 for examples illustrating this.

3.3 LCM-homomorphisms from foldings

In the coming sections, foldings will be crucial for finding embeddings of certain Artin groups into mapping class groups. We devote this section to explaining how foldings induce LCM-homomorphisms, and give a detailed proof of the main result (theorem 3.3.1) pertaining to this.

Theorem 3.3.1 (Crisp)**.** *Suppose $f : \Gamma' \to \Gamma$ is a folding. Then f induces an LCM-homomorphism $\phi^f : \mathcal{A}^+(\Gamma) \to \mathcal{A}^+(\Gamma')$ such that $\phi^f(s) \doteq \Delta_{f^{-1}(s)}$ for $s \in S$.*

Lemma 3.3.2. *Let Γ be an irreducible finite type Coxeter graph with vertex set S. Partition S into two sets K_s and K_t so that in each set, no pair of vertices are joined by an edge. If Δ_{K_s}, Δ_{K_t}, and Δ are the respective least common multiples of K_s, K_t , and $S = K_s \cup K_t$ in $\mathcal{A}^+(\Gamma)$, then $\Delta \doteq [\Delta_{K_s}, \Delta_{K_t}]$. That is, Δ is the least common multiple of Δ_{K_s} and Δ_{K_t}.*

Proof. First note that all irreducible Coxeter graphs of finite type are bipartite. Hence, one can always partition S as suggested, and the partition is actually unique up to relabeling K_s and K_t. Also note that Δ exists because Γ is of finite type, and that the existence of Δ_{K_s}, Δ_{K_t} is a consequence of lemma 5.1.1. Since Δ is a common multiple of S, Δ is a common multiple of $K_s \subset S$. Since Δ_{K_s} is the least common multiple of K_s, $\Delta_{K_s}|\Delta$. Similarly, Δ is a common multiple of K_t, and so $\Delta_{K_t}|\Delta$. Thus, Δ is a common multiple of Δ_{K_s} and Δ_{K_t}. Now suppose that C is another common multiple of Δ_{K_s} and Δ_{K_t}. Then, $C \underset{\cdot}{=} \Delta_{K_s} W$ and $C \underset{\cdot}{=} \Delta_{K_t} W'$ for some $W, W' \in \mathcal{A}^+(\Gamma)$. Since K_s and K_t consist of pairwise commuting elements and $S = K_s \cup K_t$, C is a multiple of every $s \in S$. But Δ is the least common multiple of S. So, $\Delta | C$ and therefore $\Delta \underset{\cdot}{=} [\Delta_{K_s}, \Delta_{K_t}]$. □

Lemma 3.3.3 (Brieskorn-Saito)**.** *Suppose Γ is an irreducible finite type Coxeter graph with Coxeter number h. Let $K_s = \{s_1, \cdots, s_p\}$ and $K_t = \{t_1, \cdots, t_q\}$ be a partition of S into sets of pairwise commuting generators, and let Δ be the least common multiple of S in $\mathcal{A}^+(\Gamma)$. If*

$$P' \underset{\cdot}{=} \prod_{i=1}^{p} s_i, \; P'' \underset{\cdot}{=} \prod_{j=1}^{q} t_j, \; \text{and } P \underset{\cdot}{=} P'P''$$

then

$$\begin{cases} \Delta \underset{\cdot}{=} P^{\frac{h}{2}} & \text{if } h \text{ is even} \\ \Delta \underset{\cdot}{=} P^{\frac{h-1}{2}} P' \underset{\cdot}{=} P'' P^{\frac{h-1}{2}} & \text{if } h \text{ is odd} \\ \Delta^2 \underset{\cdot}{=} P^h & \text{always} \end{cases}$$

Proof of theorem 3.3.1. We are going to show that ϕ^f satisfies conditions (1) and (2) of definition 3.2.4. Since f is surjective, $f^{-1}(s)$ is nonempty for each $s \in S$. By definition 3.2.7, it follows that, for each $s \in S$, $f^{-1}(s)$ is a finite disjoint union of vertices in Γ'. By lemma 5.2.1, $\Delta_{f^{-1}(s)}$ exists and is equal to the product (in any order) of all the elements of $f^{-1}(s)$. This takes care of condition (1) in definition 3.2.4.

For condition (2), consider $s, t \in S$ with $s \neq t$. First assume that s and t are not joined by an edge (ie $m_{st} = 2$). Since f is a simplicial map, there is no edge between any $s' \in f^{-1}(s)$ and any $t' \in f^{-1}(t)$. Hence, $\Delta_{f^{-1}(s)}$ commutes with $\Delta_{f^{-1}(t)}$. To show that $\Delta_{f^{-1}(s)}\Delta_{f^{-1}(t)} \doteq [\Delta_{f^{-1}(s)}, \Delta_{f^{-1}(t)}]$, note that $f^{-1}(s) \cup f^{-1}(t)$ consists of pairwise commuting generators each of which must divide $[\Delta_{f^{-1}(s)}, \Delta_{f^{-1}(t)}]$. This implies that $\Delta_{f^{-1}(s)}\Delta_{f^{-1}(t)}$ divides $[\Delta_{f^{-1}(s)}, \Delta_{f^{-1}(t)}]$. On the other hand, it is clear that $[\Delta_{f^{-1}(s)}, \Delta_{f^{-1}(t)}]$ divides $\Delta_{f^{-1}(s)}\Delta_{f^{-1}(t)}$. Therefore, $\Delta_{f^{-1}(s)}\Delta_{f^{-1}(t)} \doteq [\Delta_{f^{-1}(s)}, \Delta_{f^{-1}(t)}]$.

We now prove condition (2) when the f_ϵ (from definition 3.2.7) is a k-fold trivial folding. Set $f^{-1}(s) = \{s_1, \cdots, s_k\}$ and $f^{-1}(t) = \{t_1, \cdots, t_k\}$, where s and t are the vertices of ϵ. Recall that $\Delta_{f^{-1}(s)} \doteq s_1 \cdots s_k$ and $\Delta_{f^{-1}(t)} \doteq t_1 \cdots t_k$ (see first paragraph of the proof). Since s_i commutes with s_j for all i, j and s_i commutes with t_j for all $j \neq i$, it follows that

$$\begin{aligned} prod(\Delta_{f^{-1}(s)}, \Delta_{f^{-1}(t)}; m_{st}) &\doteq \prod_{i=1}^{k} prod(s_i, t_i; m_{st}) \\ &\doteq \prod_{i=1}^{k} prod(t_i, s_i; m_{st}) \\ &\doteq prod(\Delta_{f^{-1}(t)}, \Delta_{f^{-1}(s)}; m_{st}) \end{aligned}$$

Note that by definition of a k-fold trivial folding, $m_{st} = m_{s_i t_i}$ for all i. It remains to show that $prod(\Delta_{f^{-1}(s)}, \Delta_{f^{-1}(t)}; m_{st}) \doteq [\Delta_{f^{-1}(s)}, \Delta_{f^{-1}(t)}]$. Note that for each $i = 1, \cdots, k$, $[\Delta_{f^{-1}(s)}, \Delta_{f^{-1}(t)}]$ is a common multiple of $\{s_i, t_i\}$. So, $[s_i, t_i]$ divides $[\Delta_{f^{-1}(s)}, \Delta_{f^{-1}(t)}]$ for all i. Recall that $[s_i, t_i] \doteq prod(s_i, t_i, m_{st})$ (see section 3.1). So, $\{prod(s_i, t_i; m_{st})\}_{i=1}^{k}$ consists of pairwise commuting elements each of which must divide $[\Delta_{f^{-1}(s)}, \Delta_{f^{-1}(t)}]$. As such, $\prod_{i=1}^{k} prod(t_i, s_i; m_{st})$ divides $[\Delta_{f^{-1}(s)}, \Delta_{f^{-1}(t)}]$. On the other hand, it is clear that $[\Delta_{f^{-1}(s)}, \Delta_{f^{-1}(t)}]$ divides $\prod_{i=1}^{k} prod(t_i, s_i; m_{st})$. Therefore, $prod(\Delta_{f^{-1}(s)}, \Delta_{f^{-1}(t)}; m_{st}) \doteq [\Delta_{f^{-1}(s)}, \Delta_{f^{-1}(t)}]$.

Now assume that s and t are joined by an edge ϵ with label $m_{st} \geq 3$, and that the restriction f_ϵ is a dihedral folding. Since f is simplicial, $f^{-1}(\epsilon)$ is the full subgraph of Γ' spanned by its vertex set. That $f^{-1}(\epsilon)$ is irreducible follows from definition 3.2.6. Moreover, note that $\epsilon = I_2(m_{st})$ has Coxeter number m_{st}. According to definition 3.2.6, $f^{-1}(\epsilon)$ must have Coxeter number m_{st} as well. Denote the full subgraph $f^{-1}(\epsilon)$ by K, and write its bipartite partition as $K_s \cup K_t$. If m_{st} is even, then lemma 3.3.3 gives

$$\begin{aligned}
\Delta &\doteqdot P^{\frac{m_{st}}{2}} \\
&\doteqdot (\Delta_{K_s}\Delta_{K_t})^{\frac{m_{st}}{2}} \\
&\doteqdot prod(\Delta_{K_s}, \Delta_{K_t}; m_{st}) \\
&\doteqdot prod(\Delta_{K_t}, \Delta_{K_s}; m_{st})
\end{aligned}$$

where the last positive equivalence is due to the fact that lemma 3.3.3 holds irrespective of the labeling of K_s and K_t. If m_{st} is odd, then lemma 3.3.3 gives $\Delta \doteqdot P^{\frac{m_{st}-1}{2}}P' \doteqdot P''P^{\frac{m_{st}-1}{2}}$, which implies

$$\Delta \doteqdot prod(\Delta_{K_s}, \Delta_{K_t}; m_{st}) \doteqdot prod(\Delta_{K_t}, \Delta_{K_s}; m_{st})$$

By lemma 3.3.2, $prod(\Delta_{K_s}, \Delta_{K_t}; m_{st}) \doteqdot prod(\Delta_{K_t}, \Delta_{K_s}; m_{st}) \doteqdot [\Delta_{K_s}, \Delta_{K_t}]$. □

Corollary 3.3.4. *Let $\Gamma(h)$ be the Coxeter graph corresponding to an irreducible finite type Coxeter group with Coxeter number h. Then the dihedral folding of $\Gamma(h)$ onto $I_2(h)$ defines an embedding $\phi^f : \mathcal{A}^+(I_2(h)) \to \mathcal{A}^+(\Gamma(h))$ between the Artin monoids. By theorem 3.2.3, there is an embedding between the corresponding Artin groups.*

3.4 Examples of LCM-homomorphisms induced from foldings

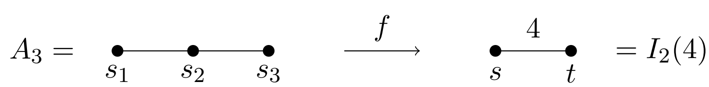
In this section, we illustrate theorem 3.3.1 by giving examples of LCM-homomorphisms between Artin monoids defined by foldings. These monoid homomorphisms induce embeddings between the corresponding Artin groups. In all the examples, notice how the Coxeter numbers match for the Coxeter graphs in the domain and range of the folding. In example 4, we show by direct computations that the map ϕ^f in theorem 3.3.1 is indeed a monoid homomorphism. This will hopefully bring about some appreciation for theorem 3.3.1. We remark that examples 4 and 5 will be used in sections 5.3 and 5.4 respectively.

1. **The dihedral foldings** $(A_3, +\epsilon)$ **and** $(A_3, -\epsilon)$ **of** A_3 **onto** $I_2(4)$**.**

   In this example, $h = 4$ (h is the Coxeter number of A_3), $K = \{s_1, s_2, s_3\}$, $\Gamma' = A_3$, and $\Gamma = \epsilon = I_2(4)$. Partition K into disjoint sets, K_s and K_t, of pairwise commuting generators. The only way to do that is by writing $K = \{s_1, s_3\} \cup \{s_2\}$. Depending on the labeling of K_s and K_t, there are two respective foldings $(A_3, +\epsilon)$ and $(A_3, -\epsilon)$, corresponding to:

   (a) $K_s = \{s_1, s_3\}$ and $K_t = \{s_2\}$.

   (b) $K_s = \{s_2\}$ and $K_t = \{s_1, s_3\}$.



   By theorem 3.3.1, the folding $(A_3, +\epsilon)$ of A_3 onto $I_2(4)$, from the first labeling, induces the LCM-homomorphism

$$\phi_+^f : \mathcal{A}^+(I_2(4)) \to \mathcal{A}^+(A_3)$$
$$s \mapsto \Delta_{f^{-1}(s)} = s_1 s_3$$
$$t \mapsto \Delta_{f^{-1}(t)} = s_2$$

   Whereas the folding $(A_3, -\epsilon)$ induces an LCM-homomorphism

$$\phi_-^f : \mathcal{A}^+(I_2(4)) \to \mathcal{A}^+(A_3)$$
$$s \mapsto \Delta_{f^{-1}(s)} = s_2$$
$$t \mapsto \Delta_{f^{-1}(t)} = s_1 s_3$$

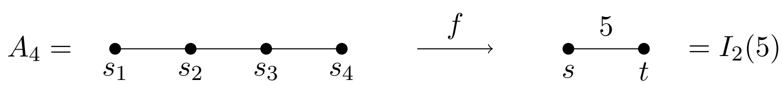
Note that $\phi^f_- = \phi^f_+ \circ \psi$, where $\psi : \mathcal{A}^+(I_2(4)) \to \mathcal{A}^+(I_2(4))$ is the homomorphism defined by $s \mapsto t$ and $t \mapsto s$. Since $\phi^f_\pm$ are LCM-homomorphisms, they are injective. By theorem 3.2.3, the induced homomorphisms between the corresponding Artin groups are injective.

2. **The dihedral folding of A_4 onto $I_2(5)$.**

In this example, $h = 5$, $K = \{s_1, s_2, s_3, s_4\}$, $\Gamma' = A_4$, and $\Gamma = \epsilon = I_2(5)$. Partition K into disjoint subsets K_s and K_t of mutually commuting elements. There is exactly one way to do that, namely $K = \{s_1, s_3\} \cup \{s_2, s_4\}$. There are two foldings $(A_4, +\epsilon)$ and $(A_4, -\epsilon)$ induced by the following labels:

(a) $K_s = \{s_1, s_3\}$ and $K_t = \{s_2, s_4\}$.

(b) $K_s = \{s_2, s_4\}$ and $K_t = \{s_1, s_3\}$.



In the first case, the (A_4, ϵ) folding f induces the LCM-homomorphism

$$\phi^f_+ : \mathcal{A}^+(I_2(5)) \to \mathcal{A}^+(A_4)$$

$$s \mapsto \Delta_{f^{-1}(s)} = s_1 s_3$$

$$t \mapsto \Delta_{f^{-1}(t)} = s_2 s_4$$

In the second case, $(A_4, -\epsilon)$ folding f induces the LCM-homomorphism

$$\phi^f_- : \mathcal{A}^+(I_2(5)) \to \mathcal{A}^+(A_4)$$

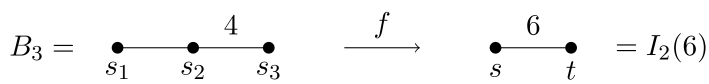
$$s \mapsto \Delta_{f^{-1}(s)} = s_2 s_4$$

$$t \mapsto \Delta_{f^{-1}(t)} = s_1 s_3$$

Notice that ϕ_+^f can be obtained from ϕ_-^f by precomposing with the isomorphism $\psi : \mathcal{A}^+(I_2(5)) \to \mathcal{A}^+(I_2(5))$ mapping s to t and t to s. Since $\phi_\pm^f$ are LCM-homomorphisms, they are injective. By theorem 3.2.3, the induced homomorphisms between the corresponding Artin groups are injective.

3. **The dihedral foldings** $(B_3, +\epsilon)$ **and** $(B_3, -\epsilon)$ **of** B_3 **onto** $I_2(6)$**.**

Here, $h = 6$, $K = \{s_1, s_2, s_3\}$, $\Gamma' = B_3$, and $\epsilon = I_2(6)$. There are two foldings $(B_3, +\epsilon)$ and $(B_3, -\epsilon)$ of B_3 onto $I_2(6)$ based on the labeling of the partition $K = \{s_1, s_3\} \cup \{s_2\}$ by K_s and K_t.



Choosing $K_s = \{s_1, s_3\}$ and $K_t = \{s_2\}$, gives the $(B_3, +\epsilon)$ folding $f : B_3 \to I_2(6)$ which, by theorem 3.3.1, induces an LCM-homomorphism

$$\phi^f : \mathcal{A}^+(I_2(6)) \to \mathcal{A}^+(B_3)$$

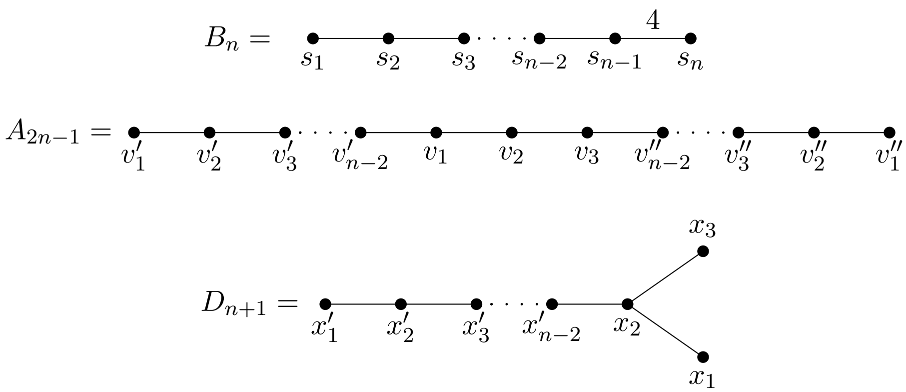
$$s \mapsto \Delta_{f^{-1}(s)} = s_1 s_3$$

$$t \mapsto \Delta_{f^{-1}(t)} = s_2$$

This monoid monomorphism induces an embedding of $\mathcal{A}(I_2(6))$ into $\mathcal{A}(B_3)$. The $(B_3, -\epsilon)$ folding, which induces a different embedding of $\mathcal{A}(I_2(6))$ into $\mathcal{A}(B_3)$, is obtained by swapping K_s and K_t.

4. **The** $(A_3, \pm\epsilon)$ **foldings of** A_{2n-1} **and** D_{n+1} **onto** B_n**.**

Consider the Coxeter graph B_n and denote by ϵ its $I_2(4)$ subgraph. Also consider the Coxeter graphs A_{2n-1} and D_{n+1} shown below



It can be verified that the simplicial map $f_+ : A_{2n-1} \to B_n$ defined by

$$\begin{aligned}
f_+(v_2) &= s_n \\
f_+(v_i) &= s_{n-1}, \;\; i = 1, 3 \\
f_+(v_j') &= s_j, \;\; j = 1, \cdots, n-2 \\
f_+(v_j'') &= s_j, \;\; j = 1, \cdots, n-2
\end{aligned}$$

is the $(A_3, +\epsilon)$ folding of A_{2n-1} onto B_n. One can also check that the simplicial map $f_- : D_{n+1} \to B_n$ defined by

$$\begin{aligned}
f_-(x_2) &= s_{n-1} \\
f_-(x_i) &= s_n, \;\; i = 1, 3 \\
f_-(x_j') &= s_j, \;\; j = 1, \cdots, n-2
\end{aligned}$$

is the $(A_3, -\epsilon)$ folding of D_{n+1} onto B_n. By theorem 3.3.1, $(A_3, +\epsilon)$ and $(A_3, -\epsilon)$ induce the LCM-homomorphisms given by

$$\phi_+^f : \mathcal{A}^+(B_n) \to \mathcal{A}^+(A_{2n-1})$$

$$\phi_+^f(s) \doteqdot \Delta_{f_+^{-1}(s)}$$

and

$$\phi_-^f : \mathcal{A}^+(B_n) \to \mathcal{A}^+(D_{n+1})$$

$$\phi_-^f(s) \doteqdot \Delta_{f_-^{-1}(s)}$$

respectively. By theorem 3.2.3, there are embeddings between the corresponding Artin groups.

We end this example by showing directly why ϕ_+^f is a monoid homomorphism. On one hand, this will shed some light on how this folding works, and on the other hand, it will help us appreciate theorem 3.3.1. From the definition of f_+, it is easily seen that

$$\begin{aligned}
\phi_+^f(s_{n-1}) &= v_1 v_3 \\
\phi_+^f(s_n) &= v_2 \\
\phi_+^f(s_j) &= v_j' v_j'',\ j = 1, \cdots, n-2
\end{aligned}$$

We show that for every relation R in $\mathcal{A}^+(B_n)$, $\phi_+^f(R)$ is a relation in $\mathcal{A}^+(A_{2n-1})$. Recall that edgeless vertices commute and that vertices joined by an unlabeled edge satisfy the braid relation.

(1) Image of $s_{n-1}s_n s_{n-1}s_n = s_n s_{n-1} s_n s_{n-1}$

$$\begin{aligned}
v_1v_3v_2v_1v_3v_2 &= v_3v_1v_2v_1v_3v_2 \\
&= v_3v_2v_1v_2v_3v_2 \\
&= v_3v_2v_1v_3v_2v_3 \\
&= v_3v_2v_3v_1v_2v_3 \\
&= v_2v_3v_2v_1v_2v_3 \\
&= v_2v_3v_1v_2v_1v_3 \\
&= v_2v_1v_3v_2v_1v_3
\end{aligned}$$

(2) Image of $s_{n-1}s_{n-2}s_{n-1} = s_{n-2}s_{n-1}s_{n-2}$

$$
\begin{aligned}
v_1 v_3 v'_{n-2} v''_{n-2} v_1 v_3 &= v_3 v_1 v'_{n-2} v''_{n-2} v_1 v_3 \\
&= v_3 v'_{n-2} v_1 v'_{n-2} v_1 v''_{n-2} v_3 \\
&= v'_{n-2} v_1 v'_{n-2} v_3 v''_{n-2} v_3 \\
&= v'_{n-2} v_1 v'_{n-2} v''_{n-2} v_3 v''_{n-2} \\
&= v'_{n-2} v''_{n-2} v_1 v'_{n-2} v_3 v''_{n-2}
\end{aligned}
$$

$$
= v'_{n-2} v''_{n-2} v_1 v_3 v'_{n-2} v''_{n-2}
$$

(3) Image of $s_n s_j = s_j s_n$ for $j = 1, \cdots, n-2$

$$
v_2 v'_j v''_j = v'_j v''_j v_2,\ j = 1, \cdots, n-2
$$

(4) Image of $s_{n-1} s_j = s_j s_{n-1}$ for $j = 1, \cdots, n-3$

$$
v_1 v_3 v'_j v''_j = v'_j v''_j v_1 v_3,\ j = 1, \cdots, n-3
$$

(5) Suppose that $j, k \in \{1, \cdots, n-2\}$ and $j \neq k$.

(a) If $m_{jk} = 2$ then $s_j s_k = s_k s_j$.

$$
v'_j v''_j v'_k v''_k = v'_k v''_k v'_j v''_j
$$

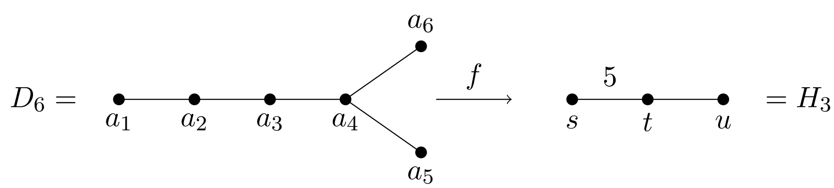
(b) If $m_{jk} = 3$ then $s_j s_k s_j = s_k s_j s_k$.

$$
\begin{aligned}
v'_j v''_j v'_k v''_k v'_j v''_j &= v''_j v'_j v'_k v'_j v''_k v''_j \\
&= v'_k v''_j v'_j v'_k v''_k v''_j \\
&= v'_k v'_j v'_k v''_j v''_k v''_j
\end{aligned}
$$

$$= v'_k v'_j v'_k v''_k v''_j v''_k$$
$$= v'_k v''_k v'_j v'_k v''_j v''_k$$
$$= v'_k v''_k v'_j v''_j v'_k v''_k$$

5. **The** (A_4, ϵ) **folding of** D_6 **onto** H_3

Consider the Coxeter graph H_3 and denote by ϵ its $I_2(5)$ subgraph.



The simplicial map $f : D_6 \to H_3$ defined by:

$$\begin{aligned} f(a_i) &= s, \ \ i = 1, 3 \\ f(a_j) &= t, \ \ j = 2, 4 \\ f(a_k) &= u, \ \ k = 5, 6 \end{aligned}$$

gives rise to the (A_4, ϵ) folding, since it restricts to the dihedral folding of A_4 over ϵ and is trivial everywhere else. By theorem 3.3.1, f induces an LCM-homomorphism defined by

$$\phi^f : \mathcal{A}^+(H_3) \to \mathcal{A}^+(D_6)$$
$$s \mapsto \Delta_{f^{-1}(s)} = a_1 a_3$$
$$t \mapsto \Delta_{f^{-1}(t)} = a_2 a_4$$
$$u \mapsto \Delta_{f^{-1}(u)} = a_5 a_6$$

By theorem 3.2.3, the induced homomorphism $\phi^f_A : \mathcal{A}(H_3) \to \mathcal{A}(D_6)$ is an embedding.

CHAPTER 4

SURFACES ASSOCIATED WITH COXETER GRAPHS AND EMBEDDED GRAPHS

4.1 Chord diagrams, curve graphs, and embedded graphs

In this section, we define a chord diagram and associate a compact orientable surface to it. We also construct graphs corresponding to chord diagrams and relate them to Coxeter graphs. Moreover, we introduce two graphs associated with a finite collection $\{a_1, \cdots, a_n\}$ of simple closed curves in an orientable surface S. The first is called the curve graph. It is a convenient way of encoding all the geometric intersections numbers $i(a_j, a_k)$. The second graph is called the embedded graph. It is simply the union $\cup_{i=1}^{n} a_i$, viewed as an embedded one-dimensional simplicial complex in S.

Definition 4.1.1. *A chord diagram in a closed disk D is a family $s_1, \cdots, s_n : [0,1] \to D$ satisfying:*

- $s_i : [0,1] \to D$ *is an embedding for all* $i \in \{1, \cdots, n\}$.
- $s_i(0), s_i(1) \in \partial D$ *for all* $i \in \{1, \cdots, n\}$.
- $s_i\left((0,1)\right) \cap \partial D = \emptyset$ *for all* $i \in \{1, \cdots, n\}$.
- *For* $i \neq j$, *either* $s_i \cap s_j = \emptyset$ *or* $s_i \cap s_j = \{x\}$, *where* $x \in intD$.
- $s_i \cap s_j \cap s_k = \emptyset$ *for distinct* i, j, *and* k.

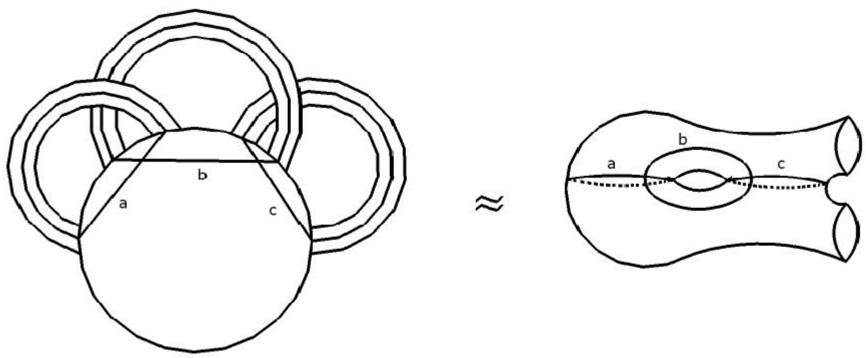


Figure 4.1: Surface defined by a chord diagram.

To a chord diagram, we associate a graph Λ whose vertices are the chords s_i, and two vertices are joined by an edge if the corresponding chords intersect. By setting $m_{ij} = 2$ when $s_i \cap s_j = \emptyset$ and $m_{ij} = 3$ when $s_i \cap s_j \neq \emptyset$, the graph Λ defines a Coxeter matrix $M = (m_{ij})$. So, we can think of Λ as a Coxeter graph whose vertex set consists of the chords s_i, and whose edges are defined above.

From a chord diagram, one can define a compact orientable surface by attaching bands to D as in Figure 4.1.

Definition 4.1.2. *Let $\{a_1, \cdots, a_n\}$ be a collection of pairwise nonisotopic simple closed curves in an orientable S. Assume that the a_i intersect efficiently in the sense of definition 1.2.1. To this collection, we associate a graph $\mathcal{CG}$, called the **curve graph** of the a_i. The vertices of $\mathcal{CG}$ are the curves a_i, and two vertices are joined by an edge whenever $i(a_i, a_j) > 0$. When $i(a_i, a_j) > 1$, the edge between a_i and a_j is labeled x_{ij} where $x_{ij} = i(a_i, a_j)$. When $i(a_i, a_j) = 1$, suppress the label.*

By setting $m_{ij} = 2$ when $i(a_i, a_j) = 0$, $m_{ij} = 3$ when $i(a_i, a_j) = 1$, and $m_{ij} = \infty$ when $i(a_i, a_j) \geq 2$, the curve graph $\mathcal{CG}$ defines a Coxeter matrix $M = (m_{ij})$. So, $\mathcal{CG}$ can be viewed as a Coxeter graph whose vertices are the simple closed curves a_i and whose edge are defined above.

Given a finite collection $\{a_1, \cdots, a_n\}$ of simple closed curves in S, the curve graph of the a_i is an efficient tool for encoding the geometric intersections $i(a_i, a_j)$.

Definition 4.1.3. *Let $\{a_1, \cdots, a_n\}$ be a collection of simple closed curves in an orientable surface S, and assume that the a_i intersect efficiently in the sense of definition 1.2.1. The* ***embedded graph*** *$\mathcal{EG}$ associated with the a_i, is $\cup_{i=1}^{n} a_i$ viewed as a one-dimensional simplicial complex embedded in S. So, the vertex set of $\mathcal{EG}$ consists of the intersection points between the a_i, and the edge set consists of the arcs joining the intersection points.*

Given an embedded graph $\mathcal{EG}$ corresponding to a finite collection $\{a_1, \cdots, a_n\}$ of simple closed curves in S, we associate to it the compact surface N_ϵ, which is a closed regular neighborhood of $\cup_{i=1}^{n} a_i$. Note that N_ϵ deformation retracts to $\mathcal{EG}$. By the homotopy invariance of the Euler characteristic, it follows that

$$\chi(N_\epsilon) = \chi(\mathcal{EG})$$

$\chi(N_\epsilon) = 2 - 2g - b$, where g represents the genus and b represents the number of boundary components of N_ϵ. Moreover, $\chi(\mathcal{EG}) = v - e$, where v and e represent the respective cardinalities of the vertex and edge sets of $\mathcal{EG}$. Hence,

$$g = \frac{e - b - v + 2}{2} \tag{4.1}$$

In subsequent sections, we shall use equation 4.1 to compute the genus of $N_\epsilon = N_\epsilon(a_1 \cup a_2 \cup a_3)$ and consequently determine its topological type.

4.2 Surfaces determined by chord diagrams corresponding to A_n and D_n

In this section, we determine the topological types of the compact surfaces associated with the chord diagrams corresponding to A_n and D_n.

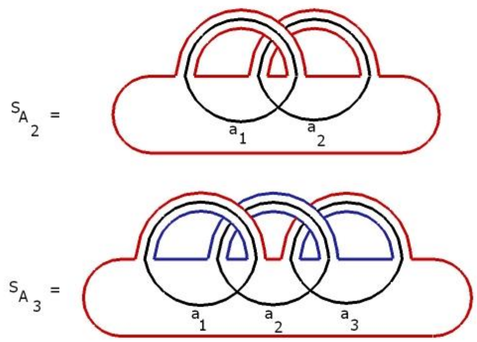


Figure 4.2: The cases $n = 2, 3$. $|\partial S_{A_2}| = 1$ and $|\partial S_{A_3}| = 2$. Distinct boundary components are shown in different colors.

Notation Let Γ be a small type Coxeter graph (ie $m_{st} \leq 3$ for all s, t) which is a tree. Let S be an arbitrary orientable surface and $a_1, \cdots, a_n$ be simple closed curves in S whose curve graph $\mathcal{CG}$ is isomorphic to Γ. Let N_ϵ be a closed regular neighborhood of $\cup_{i=1}^{n} a_i$ in S. It is very simple to check that the topological type of N_ϵ is independent of S or $\{a_i\}_{i=1}^{n}$, and only depends on Γ. This justifies using the notation S_Γ for N_ϵ. In particular, S_{A_n} (respectively S_{D_n}) shall henceforth denote a closed regular neighborhood of $\cup_{i=1}^{n} a_i$, where $\{a_1, \cdots, a_n\}$ is a collection of simple closed curves in S with curve graph A_n (respectively D_n).

Theorem 4.2.1. *Let $n \geq 2$ be an integer. If a collection $\{a_1, \cdots, a_n\}$ of simple closed curves forms an n-chain in S (ie the a_i have curve graph A_n), then*

(i) S_{A_n} is homeomorphic to $S_{\frac{n}{2},1}$ when n is even, and

(ii) S_{A_n} is homeomorphic to $S_{\frac{n-1}{2},2}$ when n is odd.

Proof. We proceed by induction on n to determine the number of boundary components of S_{A_n}. The cases $n = 2, 3$ are shown in figure 4.2. These constitute the base cases for induction when n is even and odd.

Assume that $|\partial S_{A_n}| = 1$ for some even $n \geq 4$. We shall use induction to prove $|\partial S_{A_{n+1}}| = 2$. Then, we use $|\partial S_{A_{n+1}}| = 2$ to show that $|\partial S_{A_{n+2}}| = 1$. In order to construct

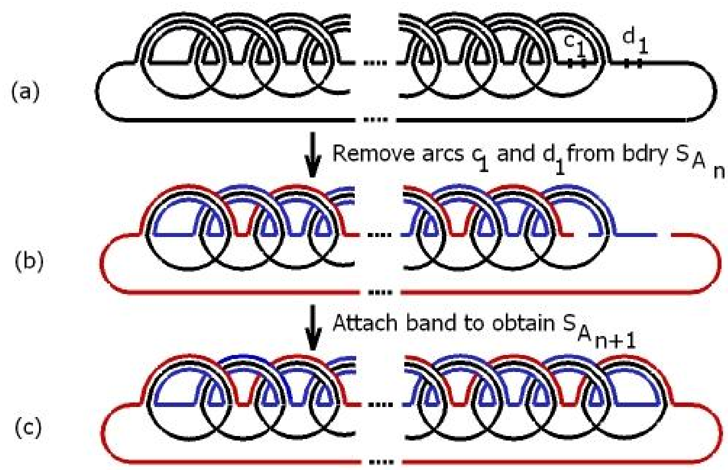


Figure 4.3: Constructing $S_{A_{n+1}}$ from S_{A_n} by attaching a band.

$S_{A_{n+1}}$ from S_{A_n}, we first remove the open arcs c_1 and d_1 from ∂S_{A_n} as shown in figure 4.3(a). What remains from ∂S_{A_n} are the two disjoint closed red and blue arcs shown in figure 4.3(b). Now attach the $(n+1)^{st}$ band to $\partial S_{A_n} \setminus \{c_1 \cup d_1\}$ as in figure 4.3(c). It can be seen in figure 4.3(c) that two colors suffice to trace the boundary of $S_{A_{n+1}}$. Thus, $|\partial S_{A_{n+1}}| = 2$.

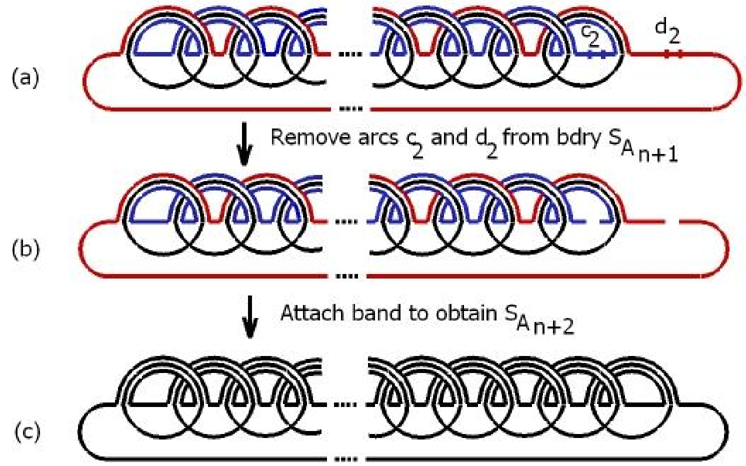


Figure 4.4: Constructing $S_{A_{n+2}}$ from $S_{A_{n+1}}$ by attaching a band.

To construct $S_{A_{n+2}}$ from $S_{A_{n+1}}$, first remove open arcs c_2 and d_2 from $\partial S_{A_{n+1}}$ as in figure 4.4(a). What remains from $\partial S_{A_{n+1}}$ are the two disjoint closed arcs in red and blue in figure 4.4(b). After attaching the $(n+2)^{nd}$ band, one can see from figure 4.4(c) that only one color suffices to trace the boundary of $S_{A_{n+2}}$. Hence, $|\partial S_{A_{n+2}}| = 1$.

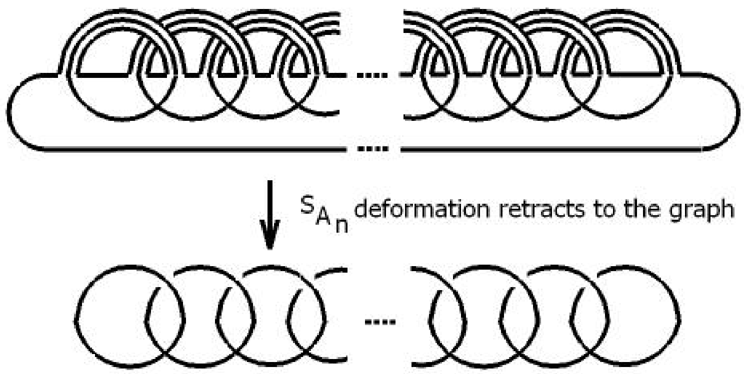


Figure 4.5: S_{A_n} deformation retracts onto its embedded graph. As such, their Euler characteristics are equal.

To determine the genus of S_{A_n}, note that it deformation retracts to the graph of figure 4.5. This graph has $n-1$ vertices and $2(n-1)$ edges. So its Euler characteristic equals $1-n$. Since the Euler characteristic is homotopy type invariant, it follows that $\chi(S_{A_n}) = 1-n$. When n is even, $2-2g-1 = 1-n$ implies $g = \frac{n}{2}$. When n is odd, $2-2g-2 = 1-n$ implies $g = \frac{n-1}{2}$. □

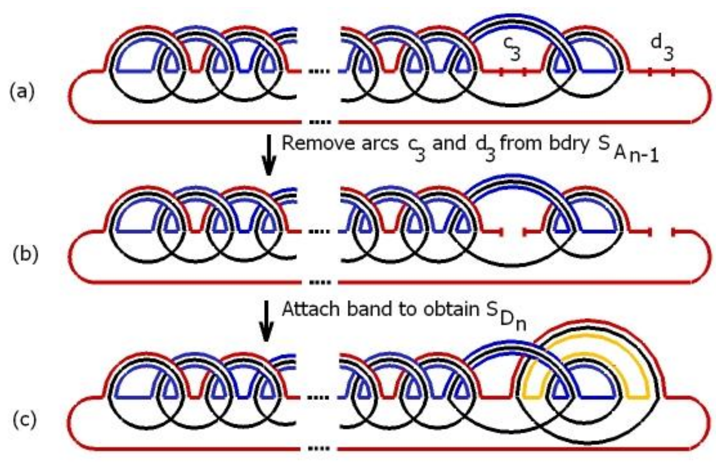


Figure 4.6: Constructing S_{D_n} from $S_{A_{n-1}}$ by attaching a band.

Theorem 4.2.2. *Let $n \geq 4$ be an integer. If a collection $\{a_1, \cdots, a_n\}$ of simple closed curves in S have curve graph D_n, then*

(i) S_{D_n} is homeomorphic to $S_{\frac{n-2}{2},3}$ when n is even, and

(ii) S_{D_n} is homeomorphic to $S_{\frac{n-1}{2},2}$ when n is odd.

Proof. Assume that n is even. By theorem 4.2.1, the subsurface $S_{A_{n-1}}$ determined by $a_1, \cdots, a_{n-1}$ has two boundary components. To construct S_{D_n} from $S_{A_{n-1}}$, first remove the open arcs c_3 and d_3 shown in figure 4.6(a). What remains from $\partial S_{A_{n-1}}$ are the two closed arcs in figure 4.6(b). Now attach the n^{th} band to obtain S_{D_n}. As shown in figure 4.6(c), three colors are needed to trace the boundary of S_{D_n}. Hence, $|\partial S_{D_n}| = 3$.

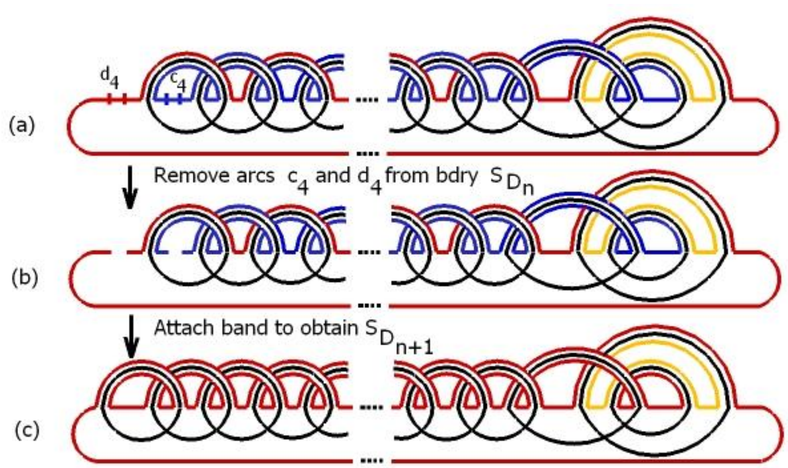


Figure 4.7: Constructing $S_{D_{n+1}}$ from S_{D_n} by attaching a band.

Using the fact that $|\partial S_{D_n}| = 3$ when n is even, we now show that $|\partial S_{D_{n+1}}| = 2$. To do so, remove arcs c_4 and d_4 from ∂S_{D_n} as shown in figure 4.7(a). What remains remains from ∂S_{D_n} are the red and blue arcs and yellow simple closed curve shown in figure 4.7(b). Now attach the $(n+1)^{st}$ band to obtain $S_{D_{n+1}}$. Figure 4.7 shows that $S_{D_{n+1}}$ has two boundary components. Finally, the genus can be found using equation 4.1. □

4.3 Surface associated with $\widetilde{A}_{n-1}$

Consider the affine Coxeter graph $\widetilde{A}_{n-1}$ defined in section 3.1. In this section, we will associate to $\widetilde{A}_{n-1}$ a compact orientable surface $\widetilde{S}$ and determine its topological type. $\widetilde{S}$ is constructed as follows. For each $i \in \{1, \cdots, n\}$, associate to the vertex s_i a compact annulus A_{s_i}, and define $\widetilde{S}$ to be the union of the annuli A_{s_i} modulo the relation $\approx$ defined as follows.

For each $j \in \{1, \cdots, n\}$ such that $m_{ij} = 3$, the relation $\approx$ identifies a square in A_{s_i} with a square in A_{s_j} so that two opposite sides of the identified square lie in ∂A_{s_i} and the other two opposite sides lie in ∂A_{s_j} (See Figure 4.8). If more than one j satisfy $m_{ij} = 3$, we stipulate that the identified squares are mutually disjoint. Note that each s_i has two neighboring vertices. More precisely, $m_{ij} = 3$ for j congruent to $i-1, i+1$ modulo n. Hence modulo n, A_{s_i} is glued to both $A_{s_{i-1}}$ and $A_{s_{i+1}}$ at disjoint squares. We restrict to gluings so that the above construction yields an orientable surface

$$\widetilde{S} := (\textstyle\coprod_{i=1}^{n} A_{s_i})/\approx$$

Pick an orientation on $\widetilde{S}$. For each annulus A_{s_i}, let $a_i : S^1 \to \widetilde{S}$ be its core curve. Choose an orientation for each a_i, and consider the union $\cup_{i=1}^{n} a_i$. This union can be viewed as an embedded graph in the surface $\widetilde{S}$. Denote this graph by $\mathcal{EG}$, and note that it has n vertices and $2n$ edges.

Denote the vertices and edges of $\mathcal{EG}$ by $v_1, \cdots, v_n$ and $e_1^+, e_1^-, \cdots, e_n^+, e_n^-$ respectively. We label the vertices and edges of $\mathcal{EG}$ as follows.

- Modulo n, $a_i = e_i^+ \cup e_i^-$.
- Modulo n, set $v_i = a_i \cap a_{i+1}$.
- Modulo n, set e_i^+ to be the edge of a_i which starts at v_{i-1} and ends at v_i, with respect to the orientation of a_i.

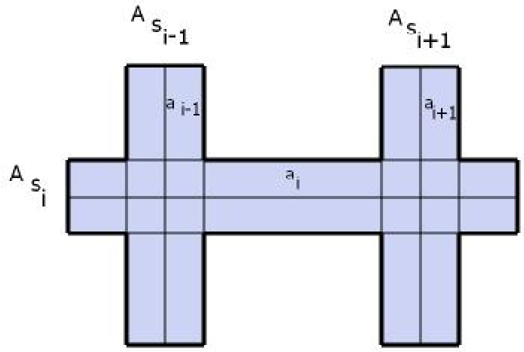


Figure 4.8: The annuli $A_{s_{i-1}}$ and $A_{s_{i+1}}$ are glued to the annulus A_{s_i} at disjoint squares.

- Modulo n, set e_i^- to be the edge of a_i which starts at v_i and ends at v_{i-1}, with respect to the orientation of a_i.

- The orientations of e_i^+ and e_i^- are induced from a_i.

For each i, consider the annulus A_{s_i} in $\widetilde{S}$. Recall that, modulo n, A_{s_i} is glued to the annuli $A_{s_{i-1}}$ and $A_{s_{i+1}}$ at two disjoint squares. By removing those squares from A_{s_i}, the closure of the complement is a disjoint union of two rectangles B_i^+ and B_i^-. To distinguish them, let B_i^+ be such that $B_i^+ \cap e_i^+ \neq \emptyset$ and B_i^- so that $B_i^- \cap e_i^- \neq \emptyset$. Now, define Re_i^+ and Le_i^+ to be the segments of $\partial B_i^+ \cap \partial\widetilde{S}$ which are to the right and left of e_i^+ respectively, with respect to the orientations of $\widetilde{S}$ and e_i^+. Similarly, define Re_i^- and Le_i^- to be the respective segments of $\partial B_i^- \cap \partial\widetilde{S}$ which are to the right and left of e_i^- (see figure 4.9 for illustration). We shall call Re_i^+, Le_i^+, Re_i^-, and Le_i^- boundary segments. Notice that

$$\partial\widetilde{S} = \cup_{i=1}^{n}(Re_i^+ \cup Le_i^+ \cup Re_i^- \cup Le_i^-)$$

As such, $\partial\widetilde{S}$ consists of $4n$ boundary segments. If we declare that each boundary segment has length 1, then the union of all the boundary segments has length $4n$. Since each boundary component of $\widetilde{S}$ must close up, we shall refer to such components as boundary cycles.

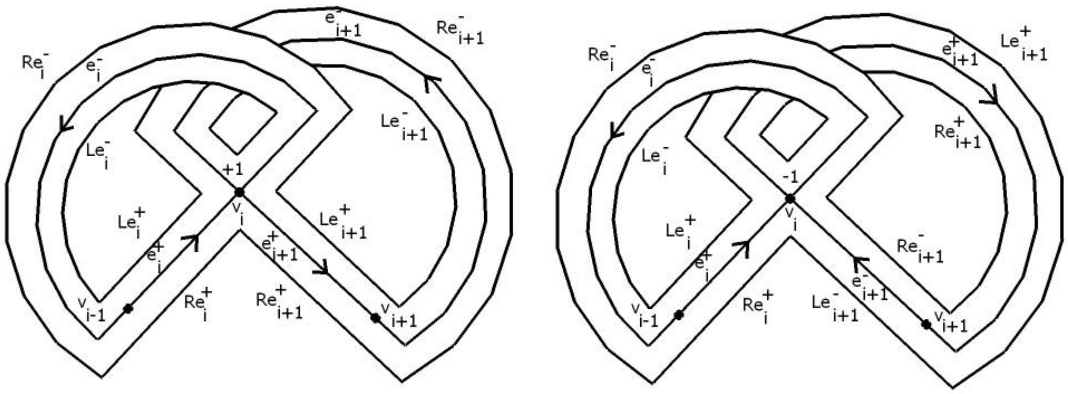


Figure 4.9: This picture illustrates the edges of $\mathcal{EG}$ around the vertex v_i when v_i has index $+1$ (left) and -1 (right). It also shows how to label the boundary segments around the simple closed curves a_i and a_{i+1}. Moreover, the proof of lemma 4.3.2 can be read out from this figure.

Definition 4.3.1. *A vertex v_i of $\mathcal{EG}$ is said to have index $+1$ if, modulo n, a_{i+1} crosses a_i from left to right at v_i. If a_{i+1} crosses a_i from right to left at v_i, then v_i is said to have index -1.*

If one starts at some boundary segment $e \in \{Re_i^+, Le_i^+, Re_i^-, Le_i^-\}$, $i = 1, \cdots, n$, and moves along a boundary path (with no backtracking), then the next boundary segment in the path depends on whether the next occurring vertex has index $+1$ or -1. The following lemma explains this fact.

Notation If e and f are two boundary segments in a boundary path of $\widetilde{S}$, write $e \to f$ to mean that f comes directly after e in the path.

Lemma 4.3.2. *Start at the boundary segment $e \in \{Re_i^+, Le_i^+, Re_i^-, Le_i^-\}$ for some $i = 1, \cdots, n$ and move towards the vertex v_i (as opposed to v_{i-1}) along a boundary path of $\widetilde{S}$. If v_i has index $+1$, then*

$$Re_i^+ \to Re_{i+1}^+$$
$$Le_i^+ \to Re_{i+1}^-$$
$$Re_i^- \to Le_{i+1}^+$$
$$Le_i^- \to Le_{i+1}^-$$

If v_i has index -1, then

$$Re_i^+ \to Le_{i+1}^-$$
$$Le_i^+ \to Le_{i+1}^+$$
$$Re_i^- \to Re_{i+1}^-$$
$$Le_i^- \to Re_{i+1}^+$$

Proof. The proof can be read out directly from Figure 4.9. □

Note that if one starts at the edge e (from lemma 4.3.2) and heads towards v_i, then the following boundary segment will track an edge of a_{i+1}. In this manner, the index of the tracked segments always increases by one modulo n.

Consider the surface $\widetilde{S} := (\coprod_{i=1}^{n} A_{s_i})/\approx$ along with the core simple closed curves a_i in A_{s_i}. Recall that $v_i = a_i \cap a_{i+1}$ modulo n. First, pick an orientation on a_1. Next, orient a_2 so that v_1 has index $+1$. Then, orient a_3 so that v_2 has index $+1$. Repeat this process for all $j = 4, \cdots, n-1$. That is, orient a_{j+1} such that v_j has index $+1$. It remains to determine the index $v_n = a_n \cap a_1$. Since the orientation on a_n was already chosen, we are not free to choose the index of v_n. This index is either $+1$ or -1 depending on how a_n intersects a_1. Based on this, there are two cases. In the first case, all the v_i have index $+1$. In the second case, $v_1, \cdots, v_{n-1}$ all have index $+1$, whereas v_n has index -1.

Definition 4.3.3. *Let $\{a_1, \cdots, a_n\}$ be a collection of simple closed curves in an orientable surface S. Assume that the curve graph of the a_i is $\widetilde{A}_{n-1}$ and set $v_i = a_i \cap a_{i+1}$ modulo n, $i = 1, \cdots, n$. We say that that the collection $\{a_1, \cdots, a_n\}$ is of **type I** if every v_i has index $+1$, and of **type II** if $v_1, \cdots, v_{n-1}$ all have index $+1$ and v_n has index -1.*

*Let N_a denote a closed regular neighborhood of $\cup_{i=1}^{n} a_i$. We say that N_a is of **type I** if $\{a_1, \cdots, a_n\}$ is of type I and N_a is of **type II** if $\{a_1, \cdots, a_n\}$ is of type II.*

Theorem 4.3.4. *Denote by b the number of boundary components of $\widetilde{S}$. If n is odd, then $\widetilde{S}$ is homeomorphic to $S_{\frac{n-1}{2},3}$. If n is even and $\widetilde{S}$ is of type I, then $\widetilde{S}$ is homeomorphic to $S_{\frac{n-2}{2},4}$. If n is even and $\widetilde{S}$ is of type II, then $\widetilde{S}$ is homeomorphic to $S_{\frac{n}{2},2}$.*

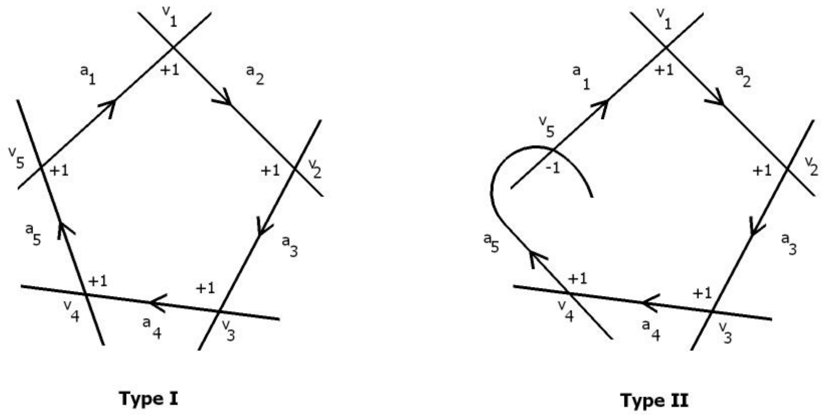


Figure 4.10: Collections of Type I and Type II.

Proof. It is easy to see that $\widetilde{S}$ deformation retracts to the graph $\mathcal{EG}$. So, $\chi(\widetilde{S}) = \chi(\mathcal{EG})$.

$$\chi(\widetilde{S}) = 2 - 2g - b$$
$$\chi(\mathcal{EG}) = n - 2n = -n$$

Hence, $g = (n - b + 2)/2$. It remains to find b. Doing that involves two cases each of which includes two sub cases. The two cases consider whether n (which is the number of vertices v_i) is odd or even. In turn, each sub case considers whether $\widetilde{S}$ is of type I or II.

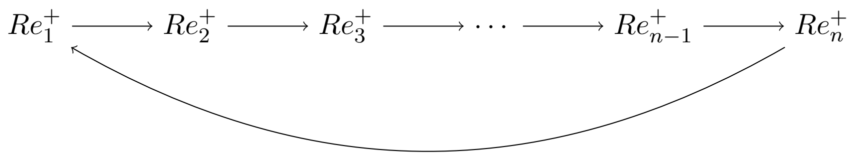
Case A. n is odd.

Case A1. $\widetilde{S}$ is of type I.

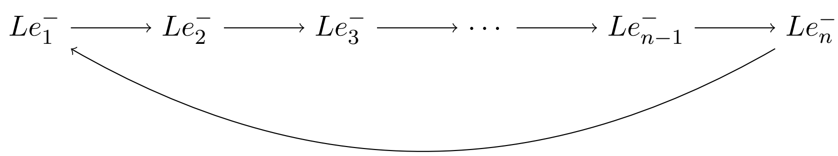
Start at Re_1^-, move towards v_1, and then follow the boundary cycle. Lemma 4.3.2 gives the following cycle C_1 of length $2n$.

$$\begin{array}{ccccccccccc}
Re_1^- & \longrightarrow & Le_2^+ & \longrightarrow & Re_3^- & \longrightarrow \cdots \longrightarrow & Le_{n-1}^+ & \longrightarrow & Re_n^- \\
\uparrow & & & & & & & & \downarrow \\
Le_n^+ & \longleftarrow & Re_{n-1}^- & \longleftarrow \cdots \longleftarrow & Le_3^+ & \longleftarrow & Re_2^- & \longleftarrow & Le_1^+
\end{array}$$

Start Re_1^+, move towards v_1, and then follow the boundary cycle. Lemma 4.3.2 gives the following cycle C_2 of length n.



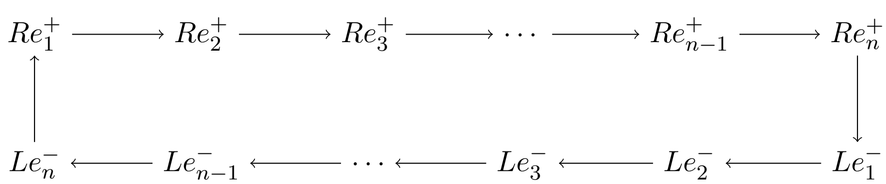
Finally, start Le_1^-, move towards v_1, and then follow the boundary cycle. By lemma 4.3.2, we have the following cycle C_3 of length n.



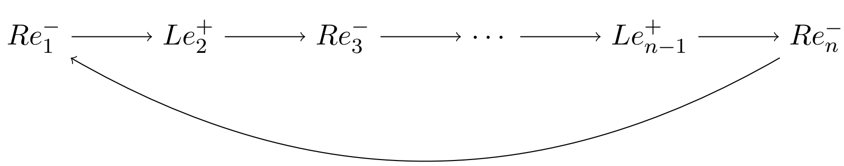
It is easy to see that C_2 and C_3 are distinct cycles. Since C_1, C_2, and C_3 are boundary cycles whose lengths add up to $4n$, they are all the boundary cycles of $\widetilde{S}$. So, $b = 3$.

Case A2. $\widetilde{S}$ is of type II.

Start at Re_1^+, move towards v_1, and then follow the boundary cycle. Lemma 4.3.2 gives the following cycle C_1 of length $2n$.



Start at Re_1^-, move towards v_1, and then follow the boundary cycle. By lemma 4.3.2, we have the following cycle C_2 of length n.



Start at Le_1^+, move towards v_1, and then follow the boundary cycle. By lemma 4.3.2, we have the following cycle C_3 of length n.

$$Le_1^+ \longrightarrow Re_2^- \longrightarrow Le_3^+ \longrightarrow \cdots \longrightarrow Re_{n-1}^- \longrightarrow Le_n^+ \ (\text{back to } Le_1^+)$$

Clearly, the cycles C_2 and C_3 are distinct. Since the C_i, $i = 1, 2, 3$ are boundary cycles whose lengths add up to $4n$, they must be all the boundary cycles of $\widetilde{S}$. Therefore, $b = 3$ in this case as well. This concludes the case when n is odd.

So the are always three boundary components when n is odd. However, in Case A1, the cycle containing Re_i^- has length $2n$ and the other two cycles have length n. In Case A2, the cycle containing Re_i^+ has length $2n$.

Case B. n is even.

Case B1. $\widetilde{S}$ is of type I.

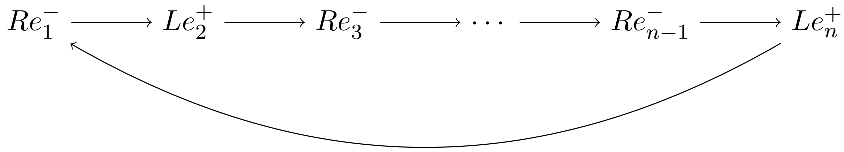
Start at Re_1^+, move towards v_1, and then follow the boundary cycle. Lemma 4.3.2 gives the following cycle C_1 of length n.

$$Re_1^+ \longrightarrow Re_2^+ \longrightarrow Re_3^+ \longrightarrow \cdots \longrightarrow Re_{n-1}^+ \longrightarrow Re_n^+ \ (\text{back to } Re_1^+)$$

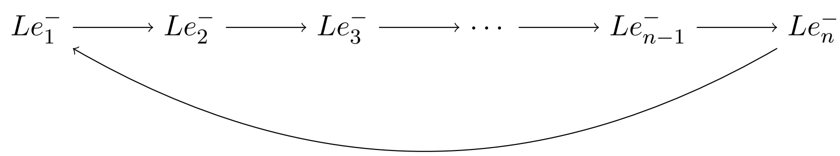
Start at Le_1^+, move towards v_1, and then follow the boundary cycle. By lemma 4.3.2, we have the following cycle C_2 of length n.

$$Le_1^+ \longrightarrow Re_2^- \longrightarrow Le_3^+ \longrightarrow \cdots \longrightarrow Le_{n-1}^+ \longrightarrow Re_n^- \ (\text{back to } Le_1^+)$$

Start at Re_1^-, move towards v_1, and then follow the boundary cycle. By lemma 4.3.2, we have the following cycle C_3 of length n.



Start at Le_1^-, move towards v_1, and then follow the boundary cycle. By lemma 4.3.2, we have the following cycle C_4 of length n.

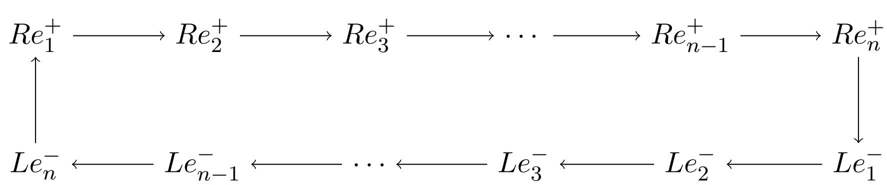


This produces four distinct boundary cycles C_i, $i = 1, 2, 3, 4$ each with length n. Since the lengths of the C_i add up to $4n$, the c_i must be all the boundary cycles of $\widetilde{S}$. As such, the number of boundary components $b = 4$.

Case B. n is even.

Case B2. $\widetilde{S}$ is of type II.

Start at Re_1^+, move towards v_1, and then follow the boundary cycle. Lemma 4.3.2 gives the following cycle C_1 of length $2n$.



Start at Re_1^-, move towards v_1, and then follow the boundary cycle. By lemma 4.3.2, we have the following cycle C_2 of length $2n$.

$$\begin{array}{ccccccccccc} Re_1^- & \longrightarrow & Le_2^+ & \longrightarrow & Re_3^- & \longrightarrow & \cdots & \longrightarrow & Re_{n-1}^- & \longrightarrow & Le_n^+ \\ \uparrow & & & & & & & & & & \downarrow \\ Re_n^- & \longleftarrow & Le_{n-1}^+ & \longleftarrow & \cdots & \longleftarrow & Le_3^+ & \longleftarrow & Re_2^- & \longleftarrow & Le_1^+ \end{array}$$

It is easy to check that C_1 and C_2 are distinct boundary cycles. Since the lengths of C_1 and C_2 add up to $4n$, they are all the boundary cycles of $\widetilde{S}$. Therefore, $b = 2$.

Finally, putting in the appropriate value of b in $g = (n - b + 2)/2$ gives the genus of $\widetilde{S}$ in each case. □

In the proof of theorem 4.3.4, it was proved that $\partial\widetilde{S}$ has three connected components (or cycles) when n is odd. Moreover, the proof shows that one of the cycles is distinguished in the sense that it has length $2n$, whereas each of the other two cycles has length n. Based on this observation, we make the following definition. This definition will be used in section 7.7.

Definition 4.3.5. *Let $n \geq 3$ be an odd integer and consider the surface $\widetilde{S}$ associated with the Coxeter graph $\widetilde{A}_{n-1}$. By theorem 4.3.4, $\widetilde{S}$ has three boundary components (or cycles) of length n, n, and $2n$. We say that a boundary component of $\widetilde{S}$ is distinguished if it has length $2n$.*

Corollary 4.3.6. *For all i, Re_i^+ and Le_i^+ belong to different boundary cycles, and Re_i^- and Le_i^- belong to different boundary cycles.*

Proof. It suffices to show the corollary for $i = 1$. From theorem 4.3.4, we can see that in Case A1, $Re_1^-, Le_1^+ \in C_1$, $Re_1^+ \in C_2$, and $Le_1^- \in C_3$. In Case A2, $Re_1^+, Le_1^- \in C_1$, $Re_1^- \in C_2$, and $Le_1^+ \in C_3$. In Case B1 $Re_1^+ \in C_1$, $Le_1^+ \in C_2$, $Re_1^- \in C_3$, and $Le_1^- \in C_4$. Finally, in Case B2, $Re_1^+, Le_1^- \in C_1$ and $Re_1^-, Le_1^+ \in C_2$. □

Corollary 4.3.7. *$\widetilde{S}$ is not homeomorphic to the surface F determined by the chord diagram of $\widetilde{A}_{n-1}$.*

Proof. It is obvious that F deformation retracts to a wedge of n circles. As such,

$$\chi(F) = 1 - n \neq -n = \chi(\widetilde{S})$$

□

CHAPTER 5

EMBEDDING ARTIN GROUPS INTO MOD(S)

5.1 Geometric homomorphisms

In this section, we discuss a natural homomorphism which relates Artin groups to mapping class groups. It is a very useful tool in the sense that it acts as a bridge between the two types of groups. It allows us to transfer interesting properties of Artin groups to $Mod(S)$.

Let $\mathcal{A}(\Gamma)$ be an Artin group of small type. That is, $m_{ij} \leq 3$ for all i, j. Let $\{a_1, \cdots, a_n\}$ be a collection of simple closed curves in S with curve graph Γ. Since Γ is of small type, no two curves in the collection intersect more than once. There is a natural homomorphism $\mathcal{A}(\Gamma) \to Mod(S)$ mapping the standard generator σ_i of $\mathcal{A}(\Gamma)$ to the (left) Dehn twist T_i along a_i. That this map is a homomorphism follows immediately from facts 1.3.7 and 1.3.8 and the definitions of a Coxeter graph and a curve graph.

In fact, it is easy to produce geometric homomorphisms from other Artin groups (not necessarily of finite type) to $Mod(S)$. Indeed, consider an arbitrary collection $\{a_1, \cdots, a_n\}$ of simple closed curves in S with curve graph $\mathcal{CG}$. Let Γ be the graph obtained from $\mathcal{CG}$ by replacing every edge label $x_{ij} \geq 2$ with ∞. It is easy to check that the map $\mathcal{A}(\Gamma) \to Mod(S)$ sending the standard generators σ_i to the Dehn twists T_i along a_i is a homomorphism. While producing homomorphisms is straightforward, the question of whether such homomorphisms are injective is quite hard.

Definition 5.1.1. *A homomorphism $\mathcal{A}(\Gamma) \to Mod(S)$ is said to be geometric if it maps the standard generators of $\mathcal{A}(\Gamma)$ to Dehn twists in $Mod(S)$. Otherwise, the homomorphism is non-geometric.*

Building on the work of Birman and Hilden [2], Perron and Vannier established the following useful result in [24]

Theorem 5.1.2 (Perron-Vannier)**.** *Let $\{a_1, \cdots, a_n\}$ be a collection of simple closed curves in S and denote by T_i the (left) Dehn twist along a_i. Suppose that the curve graph $\mathcal{CG}$ of the a_i is of type $\Gamma = A_n$ or D_n. If s_i, $i = 1, \cdots, n$ represent the standard generators of $\mathcal{A}(\Gamma)$, then the geometric homomorphism $g : \mathcal{A}(\Gamma) \to Mod(S_\Gamma)$ defined by $s_i \mapsto T_i$ is injective.*

5.2 Least common multiple lemma

Before finding explicit embeddings of Artin groups into $Mod(S)$, we prove a crucial lemma. This lemma determines the least common multiple of a finite set of mutually commuting (standard) generators in a finite type Artin monoid.

Lemma 5.2.1. *Let (W, S) be a finite type Coxeter system with Coxeter graph Γ. If $T = \{t_1, \cdots, t_k\}$ is a subset of S consisting of pairwise commuting generators, then the least common multiple ,Δ_T, of T in $\mathcal{A}^+(\Gamma)$ exists and is given by $t_{\sigma(1)}t_{\sigma(2)} \cdots t_{\sigma(k)}$, $\sigma \in \Sigma_k$*

Proof. Set $\alpha \underset{\cdot}{=} t_1 t_2 \cdots t_k$. Since the t_i pairwise commute, α is a common multiple of T. Suppose that β is another common multiple of T. Then for each $i = 1, \cdots, k$, $\exists\, x_i \in \mathcal{A}^+(\Gamma)$ such that $\beta \underset{\cdot}{=} t_i x_i$. In particular, $t_1 x_1 \underset{\cdot}{=} t_{j_1} x_{j_1}$ for all $j_1 \in \{2, \cdots, k\}$. By lemma 3.1.2 (reduction lemma), $\exists\, W_{1j_1} \in \mathcal{A}^+(\Gamma)$ such that $x_1 \underset{\cdot}{=} t_{j_1} W_{1j_1}$ for all j_1 (note that we used the assumption $m_{t_1 t_{j_1}} = 2$). In particular, $t_2 W_{12} \underset{\cdot}{=} t_{j_2} W_{1j_2}$ for all $j_2 \in \{3, \cdots, k\}$. By the reduction lemma, $\exists\, W_{12j_2} \in \mathcal{A}^+(\Gamma)$ such that $W_{12} \underset{\cdot}{=} t_{j_2} W_{12j_2}$ for all j_2. In particular, $t_3 W_{123} \underset{\cdot}{=} t_{j_3} W_{12j_3}$ for all $j_3 \in \{4, \cdots, k\}$. By the reduction lemma, $\exists\, W_{123j_3} \in \mathcal{A}^+(\Gamma)$ such that $W_{123} \underset{\cdot}{=} t_{j_3} W_{123j_3}$ for all j_3. Repeating the same process, one gets $W_{12\cdots r} \underset{\cdot}{=} t_{j_r} W_{12\cdots j_r}$ for all $j_r \in \{r+1, \cdots, k\}$, where $r \in \{4, \cdots, k\}$. In particular, when $j_1 = 2$, $j_2 = 3$, $j_3 = 4$

and $j_r = r+1$ for all $r \in \{4, \cdots, k\}$, the following equalities hold in $\mathcal{A}^+(\Gamma)$:

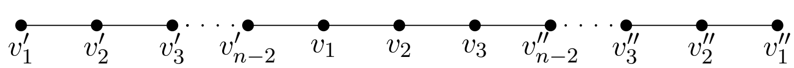
$$\begin{aligned} x_1 &\underset{\cdot}{=} t_2 W_{12} \\ W_{12} &\underset{\cdot}{=} t_3 W_{123} \\ W_{123} &\underset{\cdot}{=} t_4 W_{1234} \\ &\vdots \\ W_{12\cdots k-1} &\underset{\cdot}{=} t_k W_{12\cdots k} \end{aligned}$$

Hence, $\beta \underset{\cdot}{=} t_1 x_1 \underset{\cdot}{=} t_1 t_2 W_{12} \underset{\cdot}{=} t_1 t_2 t_3 W_{123} \underset{\cdot}{=} \cdots \underset{\cdot}{=} t_1 t_2 \cdots t_k W_{12\cdots k} \underset{\cdot}{=} \alpha W_{12\cdots k}$. Therefore, $\alpha | \beta$ and $\Delta_T \underset{\cdot}{=} \alpha$. □

5.3 Embedding $\mathcal{A}(B_n)$ into $Mod(S)$

We use the LCM-homomorphisms induced by foldings and the geometric homomorphism to give two non-geometric embeddings of $\mathcal{A}(B_n)$ into the mapping class groups $Mod(S_{A_{2n-1}})$ and $Mod(S_{D_{n+1}})$.

Theorem 5.3.1. *Let $n \geq 3$ be an integer. Suppose that curves $v'_1, v'_2, \cdots, v'_{n-2}, v_1, v_2, v_3, v''_{n-2}, v''_2, \cdots, v''_1$ form a $(2n-1)$-chain in $S_{A_{2n-1}}$ (recall that $S_{A_{2n-1}}$ is a closed regular neighborhood of the union of these curves). Then the curve graph is*

$A_{2n-1} =$ 

If T_i, T'_i, and T''_i denote the respective (left) Dehn twists along v_i, v'_i, and v''_i, then the subgroup G of $Mod(S_{A_{2n-1}})$ generated by the set $\{T'_j T''_j, T_1 T_3, T_2 \mid j = 1, \cdots, n-2\}$ is isomorphic to $\mathcal{A}(B_n)$. More precisely, if we set $\sigma_j = T'_j T''_j$ for $j = 1, \cdots, n-2$, $\sigma_{n-1} = T_1 T_3$, and $\sigma_n = T_2$, then

$$\begin{aligned} G \;=\; \langle \sigma_1, \cdots, \sigma_n | \; & \sigma_j \sigma_{j+1} \sigma_j = \sigma_{j+1} \sigma_j \sigma_{j+1} \; for \;\; j = 1, \cdots, n-2 \\ & \sigma_j \sigma_k = \sigma_k \sigma_j \; for \; |j-k| \geq 2, \;\; \sigma_{n-1} \sigma_n \sigma_{n-1} \sigma_n = \sigma_n \sigma_{n-1} \sigma_n \sigma_{n-1} \rangle \end{aligned}$$

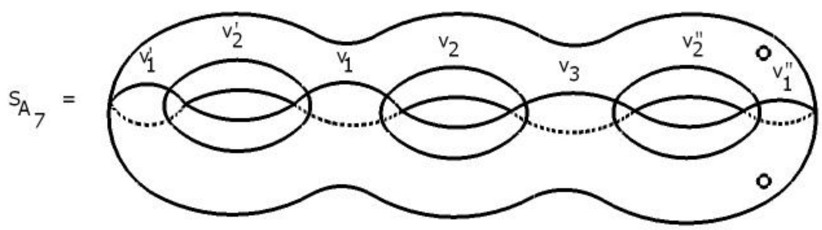


Figure 5.1: This picture illustrates theorem 5.3.1 when $n = 4$. By theorem 4.2.1, S_{A_7} is homeomorphic to $S_{3,2}$. The subgroup of $Mod(S_{3,2})$ generated by $T_1'T_1''$, $T_2'T_2''$, T_1T_3 and T_2 is isomorphic to $\mathcal{A}(B_4)$.

Proof. As shown in example 4 of section 3.4, the $(A_3, +\epsilon)$ folding of A_{2n-1} onto B_n induces the LCM-homomorphism

$$\phi^f : \mathcal{A}^+(B_n) \to \mathcal{A}^+(A_{2n-1})$$

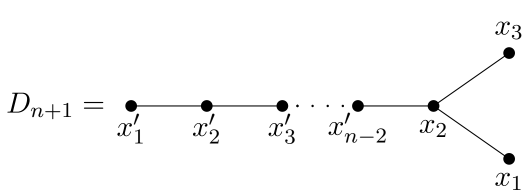
$$s_j \mapsto \Delta_{f^{-1}(s_j)}, \quad j = 1, \cdots, n-2$$

$$s_{n-1} \mapsto \Delta_{f^{-1}(s_{n-1})}$$

$$s_n \mapsto \Delta_{f^{-1}(s_n)}$$

By Lemma 5.2.1, $\Delta_{f^{-1}(s_n)} = v_2$, $\Delta_{f^{-1}(s_{n-1})} = v_1v_3$, and $\Delta_{f^{-1}(s_j)} = v_j'v_j''$. By Theorem 3.2.3, ϕ^f induces an embedding $\phi : \mathcal{A}(B_n) \to \mathcal{A}(A_{2n-1})$. Theorem 5.1.2 implies that the geometric homomorphism $g : \mathcal{A}(A_{2n-1}) \to Mod(S_{A_{2n-1}})$ is injective. Hence, $g \circ \phi$ is an isomorphism of $\mathcal{A}(B_n)$ onto its image. This image is the subgroup of $Mod(S_{A_{2n-1}})$ generated by $\{T_2, T_1T_3, T_j'T_j'' \mid j = 1, \cdots, n-2\}$. □

Theorem 5.3.2. *Let $n \geq 3$ be an integer. Suppose that curves $x_1', x_2', \cdots, x_{n-2}', x_1, x_2, x_3$ have curve graph*



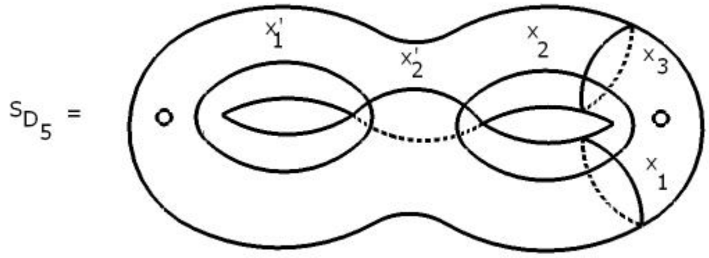


Figure 5.2: This picture illustrates theorem 5.3.2 when $n = 4$. By theorem 4.2.2, S_{D_5} is homeomorphic to $S_{2,2}$. The subgroup of $Mod(S_{2,2})$ generated by T_1', T_2', T_2, and T_1T_3 is isomorphic to $\mathcal{A}(B_4)$.

If T_i and T_i' represent the respective (left) Dehn twists along x_i and x_i', then the subgroup G of $Mod(S_{D_{n+1}})$ generated by the set $\{T_j', T_2, T_1T_3 \mid j = 1, \cdots, n-2\}$ is isomorphic to $\mathcal{A}(B_n)$. Specifically, if we set $\sigma_j = T_j'$ for $j = 1, \cdots, n-2$, $\sigma_{n-1} = T_2$, and $\sigma_n = T_1T_3$, then

$$\begin{aligned} G \;=\; & \langle \sigma_1, \cdots, \sigma_n \mid \sigma_j\sigma_{j+1}\sigma_j = \sigma_{j+1}\sigma_j\sigma_{j+1} \; for \;\; j = 1, \cdots, n-2 \\ & \sigma_j\sigma_k = \sigma_k\sigma_j \; for \; |j-k| \geq 2, \;\; \sigma_{n-1}\sigma_n\sigma_{n-1}\sigma_n = \sigma_n\sigma_{n-1}\sigma_n\sigma_{n-1} \rangle \end{aligned}$$

Proof. It follows from example 4 of section 3.4 that the $(A_3, -\epsilon)$ folding of D_{n+1} onto B_n induces the LCM-homomorphism

$$\phi^f : \mathcal{A}^+(B_n) \to \mathcal{A}^+(D_{n+1})$$

$$s_n \mapsto \Delta_{f^{-1}(s_n)} = x_1x_3$$

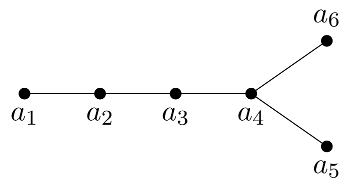
$$s_{n-1} \mapsto \Delta_{f^{-1}(s_{n-1})} = x_2$$

$$s_j \mapsto \Delta_{f^{-1}(s_j)} = x_j', \quad j = 1, \cdots, n-2$$

ϕ^f induces a monomorphism ϕ between the corresponding Artin groups. Since the geometric homomorphism $g : \mathcal{A}(D_{n+1}) \to Mod(S_{D_{n+1}})$ is injective, $g \circ \phi : \mathcal{A}(B_n) \to G$ is a geometric isomorphism. As such, G has the desired presentation. □

5.4 Embedding $\mathcal{A}(H_3)$ into $Mod(S)$

Theorem 5.4.1. *Suppose that curves $a_1, \cdots, a_6$ in $S_{D_6} = S_{2,3}$ have curve graph*

$D_6 =$ 

Then the subgroup G of $Mod(S_{2,3})$ generated by T_1T_3, T_2T_4, and T_5T_6 is isomorphic to the Artin group $\mathcal{A}(H_3)$. More precisely, if we set $\sigma_1 = T_1T_3$, $\sigma_2 = T_2T_4$, and $\sigma_3 = T_5T_6$, then

$$G = \langle \sigma_1, \sigma_2, \sigma_3 | \, \sigma_1\sigma_2\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2\sigma_1\sigma_2 \, , \sigma_1\sigma_3 = \sigma_3\sigma_1 \, , \, \sigma_2\sigma_3\sigma_2 = \sigma_3\sigma_2\sigma_3 \rangle$$

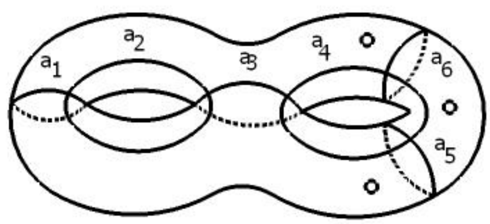
$S_{D_6} =$ 

Figure 5.3: This picture illustrates theorem 5.4.1. By theorem 4.2.2, S_{D_6} has topological type $S_{2,3}$. The subgroup of $Mod(S_{2,3})$ generated by T_1T_3, T_2T_4, and T_5T_6 is isomorphic to $\mathcal{A}(H_3)$.

Proof. As seen in example 5 section 3.4, the (A_4, ϵ) folding of D_6 onto H_3 induces the LCM-homomorphism

$$\phi^f : \mathcal{A}^+(H_3) \to \mathcal{A}^+(D_6)$$

$$s \mapsto \Delta_{f^{-1}(s)} = a_1a_3$$

$$t \mapsto \Delta_{f^{-1}(t)} = a_2a_4$$

$$u \mapsto \Delta_{f^{-1}(u)} = a_5a_6$$

The composition $g \circ \phi$ of the induced Artin group homomorphism $\phi : \mathcal{A}(H_3) \to \mathcal{A}(D_6)$ and the geometric homomorphism $g : \mathcal{A}(D_6) \to Mod(S_{2,3})$ gives the desired isomorphism. $\square$

5.5 $\mathcal{A}(\widetilde{A}_{n-1})$ as a subgroup of $Mod(S)$

In this section, we study isomorphic images of the affine Artin group $\mathcal{A}(\widetilde{A}_{n-1})$ in $Mod(S)$. First, we find a geometric embedding of $\mathcal{A}(\widetilde{A}_{n-1})$ into $Mod(S_{A_n})$. From this, we obtain a geometric embedding of $\mathcal{A}(\widetilde{A}_{n-1})$ into $Mod(\widetilde{S})$, where $\widetilde{S}$ is the surface defined in section 4.3. Next, we prove a fundamental lemma (lemma 5.5.3) which states that given two collections $\{a_1, \cdots, a_n\}$ and $\{a'_1, \cdots, a'_n\}$ of simple closed curves with curve graphs $\widetilde{A}_{n-1}$, the subgroups G of $Mod(N_a)$ and G' of $Mod(N'_a)$ generated by $\{T_i\}$ and $\{T'_i\}$, $i = 1, \cdots, n$ respectively, are isomorphic provided that the closed regular neighborhoods N_a of $\cup_{i=1}^n a_i$ and N'_a of $\cup_{i=1}^n a'_i$ are homeomorphic. We illustrate the fundamental lemma with the specific example when $n = 3$. Finally, we prove that a subgroup of $Mod(\widetilde{S})$ generated by Dehn twists along simple closed curves with curve graph $\widetilde{A}_{n-1}$ is isomorphic to $\mathcal{A}(\widetilde{A}_{n-1})$, whenever n is odd or n is even and $\widetilde{S}$ is of type II.

Theorem 5.5.1. *Consider a chain $b_1, \cdots, b_n$ of simple closed curves in an orientable surface S and denote by N_b a closed regular neighborhood of $\cup_{i=1}^n b_i$. Let T_i be the (left) Dehn twist along b_i and set $\alpha = T_1^2 T_2 \cdots T_{n-1}(b_n)$. If $\mathcal{A}(\widetilde{A}_{n-1})$ has standard generators $\sigma_1, \cdots, \sigma_n$, then the homomorphism $\phi : \mathcal{A}(\widetilde{A}_{n-1}) \to Mod(N_b)$ defined by*

$$\sigma_j \mapsto T_{j+1},\ j = 1, \cdots, n-1$$

$$\sigma_n \mapsto T_\alpha$$

is a geometric embedding. Moreover, the collection $\{b_2, \cdots, b_n, \alpha\}$ has curve graph $\widetilde{A}_{n-1}$.

Proof. The classical braid group $\mathcal{B}_{n+1}$ on $n+1$ strands is isomorphic to the Artin group $\mathcal{A}(A_n)$. The isomorphism $\mathcal{A}(A_n) \cong \mathcal{B}_{n+1}$ is given by mapping the standard generator γ_i to the braid where the i^{th} strand crosses over the $(i+1)^{st}$. For simplicity, we shall denote this braid (ie the image of γ_i) by γ_i as well.

$$\mathcal{A}(A_n) = \langle \gamma_1, \cdots, \gamma_n \mid \gamma_i\gamma_{i+1}\gamma_i = \gamma_{i+1}\gamma_i\gamma_{i+1}, \gamma_i\gamma_j = \gamma_j\gamma_i\ if\ |i-j| \geq 2 \rangle$$

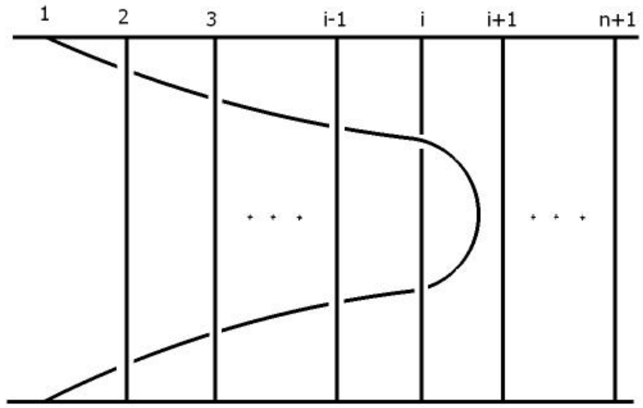


Figure 5.4: a_j is a generator of D_{n+1}. In terms of the standard generators of $\mathcal{B}_{n+1}$, $a_j = \gamma_1^{-1}\gamma_2^{-1}\cdots\gamma_{j-2}^{-1}\gamma_{j-1}^2\gamma_{j-2}\cdots\gamma_2\gamma_1$.

Since D_{n+1} is a subgroup of $\mathcal{B}_{n+1}$, the generators of D_{n+1} can be expressed in terms of the standard generators $\gamma_1, \cdots, \gamma_n$ of $\mathcal{B}_{n+1}$. D_{n+1} is generated by $\gamma_2, \cdots, \gamma_n$ and $a_2, \cdots, a_{n+1}$ (see section 2.3). The inclusion monmomorphism $i_2 : D_{n+1} \hookrightarrow \mathcal{B}_{n+1}$ is given by

$$\gamma_i = \gamma_i \text{ for } i = 2, \cdots, n$$

$$a_2 = \gamma_1^2$$

$$a_j = (\gamma_{j-2}\cdots\gamma_2\gamma_1)^{-1}\gamma_{j-1}^2(\gamma_{j-2}\cdots\gamma_2\gamma_1) \text{ for } j = 3, \cdots, n+1$$

The inverse $\Phi^{-1} : CB_n \to D_{n+1}$ of the isomorphism Φ is given by:

$$\sigma_{n-i} \mapsto \gamma_i \text{ for } i = 2, \cdots, n$$

$$\sigma_{n-1} \mapsto (a_2\gamma_2\cdots\gamma_n)\gamma_n(a_2\gamma_2\cdots\gamma_n)^{-1}$$

$$\tau \mapsto (a_2\gamma_2\cdots\gamma_n)^{-1}$$

$$(\sigma_{n-2}\cdots\sigma_0\tau)^{-1} \mapsto a_2$$

$$(\sigma_{(n+1)-j}\cdots\sigma_{n-2})(\sigma_{n-2}\sigma_{n-3}\cdots\sigma_0\tau)^{-1}(\sigma_{(n+1)-j}\cdots\sigma_{n-2})^{-1} \mapsto a_j$$

$$\text{for } j = 3, \cdots, n+1$$

Note that the b_i have curve graph A_n. Hence, N_b is homeomorphic to S_{A_n}, which by theorem 4.2.1 has topological type $S_{\frac{n}{2},1}$ when n is even and $S_{\frac{n-1}{2},2}$ when n is odd.

By theorem 5.1.2 (Perron-Vannier) , the homomorphism $g : \mathcal{A}(A_n) \to Mod(N_b)$ given by $\gamma_i \mapsto T_i$ is injective. As such, the following composition of monomorphisms gives an explicit geometric embedding of $\mathcal{A}(\widetilde{A}_{n-1})$ into $Mod(N_b)$. Recall that N is the normal subgroup of CB_n generated by $\sigma_0, \cdots, \sigma_{n-1}$ (see section 2.3). It was shown in section 2.3 that N and $\mathcal{A}(\widetilde{A}_{n-1})$ are isomorphic.

$$N \stackrel{i_1}{\hookrightarrow} CB_n \stackrel{\Phi^{-1}}{\to} D_{n+1} \stackrel{i_2}{\hookrightarrow} \mathcal{B}_{n+1} \stackrel{g}{\hookrightarrow} Mod(N_b)$$

$$\sigma_{n-i} = \sigma_{n-i} \mapsto \gamma_i = \gamma_i \mapsto T_i,\ i = 2, \cdots, n$$

$$\sigma_{n-1} = \sigma_{n-1} \to (a_2\gamma_2 \cdots \gamma_n)\gamma_n(a_2\gamma_2 \cdots \gamma_n)^{-1} \mapsto \gamma_1^2\gamma_2 \cdots \gamma_{n-1}\gamma_n\gamma_{n-1}^{-1} \cdots \gamma_2^{-1}\gamma_1^{-2} \mapsto$$
$$T_1^2T_2 \cdots T_{n-1}T_nT_{n-1}^{-1} \cdots T_2^{-1}T_1^{-2} = T_{T_1^2T_2 \cdots T_{n-1}(b_n)}$$

Set $\phi := g \circ i_2 \circ \Phi^{-1} \circ i_1$. The map ϕ is a geometric embedding, and so $\phi(N)$ is a subgroup of $Mod(N_b)$ which is isomorphic to $\mathcal{A}(\widetilde{A}_{n-1})$. This subgroup is generated by the (left) Dehn twists along the curves $b_2, \cdots, b_n$, and $\alpha = T_1^2T_2 \cdots T_{n-1}(b_n)$.

It remains to show that $\{b_2, \cdots, b_n, \alpha\}$ has curve graph $\widetilde{A}_{n-1}$. Since ϕ is a geometric homomorphism, it follows that

$$T_nT_\alpha T_n = T_\alpha T_n T_\alpha$$
$$T_2T_\alpha T_2 = T_\alpha T_2 T_\alpha$$
$$T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1} \text{ for } i = 2, \cdots, n-1$$
$$T_iT_j = T_jT_i \text{ for } |i-j| \geq 2$$
$$T_\alpha T_k = T_kT_\alpha \text{ for } k = 3, \cdots, n-1$$

By facts 1.3.7 and 1.3.8, the curve graph associated with the curves $b_2, \cdots, b_n$ and α is isomorphic to $\widetilde{A}_{n-1}$. □

Denote by S' a closed regular neighborhood of $(\cup_{j=2}^n b_j) \cup \alpha$. Hence, S' is a subsurface of N_b which is homeomorphic to $\widetilde{S}$. In proposition 5.5.2, we show that the subgroup of $Mod(S')$ generated by $\{T_2, \cdots, T_n, T_\alpha\}$ is isomorphic to $\mathcal{A}(\widetilde{A}_{n-1})$.

Proposition 5.5.2. *Let G be the subgroup of $Mod(S')$ generated by $T_2, \cdots, T_n$ and T_α. Then $G \cong \mathcal{A}(\widetilde{A}_{n-1})$.*

Proof. The inclusion of S' as a subsurface of N_b induces the natural homomorphism $i_* : Mod(S') \to Mod(N_b)$ defined by extending by the identity of the compliment. It is immediate from the definition of i_* that $i_*(G)$ is the subgroup of $Mod(N_b)$ generated by $T_1, \cdots, T_n$. By theorem 5.5.1 $i_*(G) \cong \mathcal{A}(\widetilde{A}_{n-1})$.

Denote the restriction of i_* to G by i_* as well. This restriction yields an epimorphism $i_* : G \to i_*(G)$. Define $h : i_*(G) \to G$ by $h(T_i) = T_i$. Since the images of all the defining relations in $i_*(G)$ are satisfied in G, h is well-defined map which is clearly a homomorphism. Since the T_i generate G, h is surjective. Since $h \circ i_* = 1_G$ and $i_* \circ h = 1_{i_*(G)}$, $h = (i_*)^{-1}$. Therefore, $G \cong \mathcal{A}(\widetilde{A}_{n-1})$. □

Proposition 5.5.2 provides a collection $\mathcal{C} = \{b_2, \cdots, b_n, \alpha\}$ of simple closed curves in $\widetilde{S}$ such that $\mathcal{C}$ has curve graph $\widetilde{A}_{n-1}$, and the subgroup of $Mod(\widetilde{S})$ generated by the (left) Dehn twists $T_2, \cdots, T_n$, and T_α is isomorphic to $\mathcal{A}(\widetilde{A}_{n-1})$.

Lemma 5.5.3. *Let S be an orientable surface, and consider two collections $\{a_1, \cdots, a_n\}$ and $\{a'_1, \cdots, a'_n\}$ of simple closed curves in S with curve graphs $\widetilde{A}_{n-1}$. Modulo n, set $v_i = a_i \cap a_{i+1}$, $i = 1, \cdots, n$ (respectively $v'_i = a'_i \cap a'_{i+1}$), and let T_i (respectively T'_i) represent the (left) Dehn twist along a_i (respectively a'_i). Denote by N_a and N'_a closed regular neighborhoods of $\cup_{i=1}^n a_i$ and $\cup_{i=1}^n a'_i$ respectively. Denote by G and G' the respective subgroups of $Mod(N_a)$ and $Mod(N'_a)$ that are generated by $T_1, \cdots, T_n$ and $T'_1, \cdots, T'_n$.*

(i) If N_a is homeomorphic to N'_a and they have the same type (see definition 4.3.3), then there exists an orientation preserving homeomorphism $\widetilde{f} : N_a \to N'_a$ such that $\widetilde{f}(a_i) = a'_i$.

(ii) If N_a is homeomorphic to N'_a and they have distinct types, then there exists an orientation preserving homeomorphism $\widetilde{f} : N_a \to N'_a$ such that $\widetilde{f}(a_i) = a'_i$ when i is even and $\widetilde{f}(a_i) = (a'_i)^{-1}$ when i is odd.

In both cases, $\widetilde{f}$ induces an isomorphism $G \cong G'$ given by $T_i \mapsto T'_i$.

Proof. First assume that n is odd. There are three cases to consider:

1. N_a and N'_a are both of type I.

2. N_a and N'_a are both of type II.

3. N_a is of type I and N'_a is of type II.

By theorem 4.3.4, N_a and N'_a are homeomorphic in all three cases. We must find an orientation preserving homeomorphism $\widetilde{f} : N_a \to N'_a$ satisfying $\widetilde{f}(a_i) = a'_i$. To do that, we first define a homeomorphism $f : \partial N_a \to \partial N'_a$, then extend it to obtain $\widetilde{f}$.

In the first two cases, let $f : \partial N_a \to \partial N'_a$ be a homeomorphism which maps each segment in ∂N_a to its analogous segment in $\partial N'_a$. That is, $Re_i^{\pm} \mapsto R(e_i^{\pm})'$ and $Le_i^{\pm} \mapsto L(e_i^{\pm})'$ for all $i = 1, \cdots, n$. Note that the boundary segments match in the sense that $f(Re_i^{\pm})$ and $f(Le_i^{\pm})$ always track the edge $(e_i^{\pm})'$ in $\mathcal{EG}' = \cup_{i=1}^{n} a'_i$. This allows us to extend f to the rectangles B_i^+ and B_i^- defined in section 4.3. Also notice that whenever two boundary segments in ∂N_a intersect at one point, their images under f intersect at one point as well. This together with the sides matching allow us to extend f to a homeomorphism on a regular neighborhood of ∂N_a. To see this, consider any eight boundary segments β_i, β'_i,

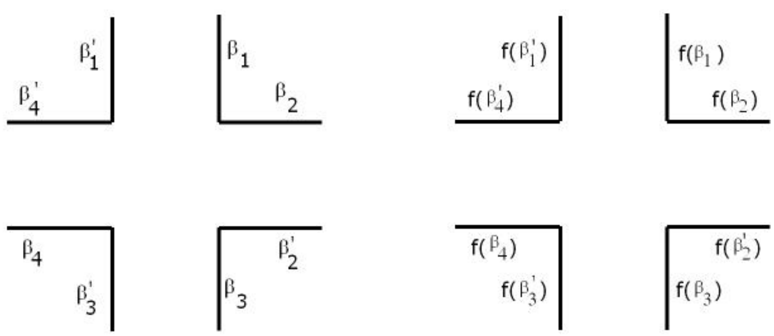


Figure 5.5: The homeomorphism $f : \partial N_a \to \partial N'_a$ maps matching segments to matching segments and intersecting segments to intersecting segments. This allows us to extend f to a homeomorphism $\tilde{f}$ on a regular neighborhood of ∂N_a.

$i = 1, 2, 3, 4$, around an identified square in N_a so that, for each i, β_i and β'_i track an edge in $\mathcal{EG} = \cup_{i=1}^{n} a_i$, and for each pair $\{\beta_1, \beta_2\}$, $\{\beta'_2, \beta_3\}$, $\{\beta'_3, \beta_4\}$, $\{\beta'_4, \beta'_1\}$ the two segments intersect at one point (see figure 5.5). Since β_1 and β_1' track an edge in $\mathcal{EG}$, $f(\beta_1)$ and $f(\beta_1')$ track an edge in $\mathcal{EG}'$. Since β_1 and β_2 intersect at one point, so must $f(\beta_1)$ and $f(\beta_2)$. Since β_2 and β_2' track an edge in $\mathcal{EG}$, $f(\beta_2)$ and $f(\beta_2')$ track an edge in $\mathcal{EG}'$. Since β_2' and β_3 intersect at one point, so must $f(\beta_2')$ and $f(\beta_3)$. Since β_3 and β_3' track an edge in $\mathcal{EG}$, $f(\beta_3)$ and $f(\beta_3')$ track an edge in $\mathcal{EG}'$. Since β_2' and β_4 intersect at one point, so must $f(\beta_3')$ and $f(\beta_4)$. Since β_4 and β_4' track an edge in $\mathcal{EG}$, $f(\beta_4)$ and $f(\beta_4')$ track an edge in $\mathcal{EG}'$. Finally, $f(\beta_4')$ and $f(\beta_1)$ intersect at one point since β_4' and β_1 do. As such, one can extend f to a regular neighborhood of $\cup_{i=1}^4(\beta_i \cup \beta_i')$.

So far, we have extended $f : \partial N_a \to N_a'$ to all the rectangles $B_i^{\pm}$ and to a regular neighborhood of ∂N_a. What remains is a disjoint union of disks. Extend f uniquely on those disks, and therefore on all of N_a. Now that we have a homeomorphism $\widetilde{f} : N_a \to N_a'$, orient N_a' so that $\widetilde{f}$ is orientation preserving. To assure that this happens, simply pick the appropriate orientation on N_a'.

$\widetilde{f}$ induces a homeomorphism $\mathcal{EG} \to \mathcal{EG}'$ between the embedded graphs. This homeomorphism is given by:

$$\mathcal{EG} \ni e_i^+ \mapsto (e_i^+)' \in \mathcal{EG}'$$
$$\mathcal{EG} \ni e_i^- \mapsto (e_i^-)' \in \mathcal{EG}'$$

Since $a_i = e_i^+ \cup e_i^-$ and $a_i' = (e_i^+)' \cup (e_i^-)'$, $\widetilde{f}(a_i) = a_i'$ for each $i = 1, \cdots, n$.

The third case is much more interesting. We would like to map ∂N_a to $\partial N_a'$ so that the images of matching boundary segments in ∂N_a match in $\partial N_a'$. Moreover, whenever two boundary segments intersect at one point in ∂N_a, we want the corresponding image segments to do the same.

Consider the distinguished boundary cycle C_1 of ∂N_a and the distinguished boundary cycle C_1' of $\partial N_a'$. Each of those cycles has length $2n$. As seen in the proof of theorem 4.3.4, C_1 starts with Re_1^- while C_1' begins with $R(e_1^+)'$. Map Re_1^- in C_1 to $R(e_1^+)'$ in C_1'. One must then map Le_1^- (in C_3) (see proof of theorem 4.3.4) to $L(e_1^+)'$ (in C_3') so that the boundary segments in the image match. Now map the second segment Le_2^+ in C_1 to the second segment $R(e_2^+)'$ in C_1'. Then, Re_2^+ (in C_2) must map to $L(e_2^+)'$ (in C_2'). Continue this process while following all the segments of C_1. This gives a homeomorphism $f : \partial N_a \to \partial N_a'$ defined by

$$Re_i^- \mapsto \begin{cases} R(e_i^+)' & \text{when } i \text{ is odd} \\ L(e_i^-)' & \text{when } i \text{ is even} \end{cases}$$

$$Le_i^+ \mapsto \begin{cases} L(e_i^-)' & \text{when } i \text{ is odd} \\ R(e_i^+)' & \text{when } i \text{ is even} \end{cases}$$

$$Re_i^+ \mapsto \begin{cases} R(e_i^-)' & \text{when } i \text{ is odd} \\ L(e_i^+)' & \text{when } i \text{ is even} \end{cases}$$

$$Le_i^- \mapsto \begin{cases} L(e_i^+)' & \text{when } i \text{ is odd} \\ R(e_i^-)' & \text{when } i \text{ is even} \end{cases}$$

It can be easily checked that the boundary segments match and whenever two boundary segments intersect at one point, their corresponding images under f intersect at one point as well. So, f extends to a homeomorphism $\widetilde{f} : N_a \to N_a'$. Pick the appropriate orientation on N_a' so that $\widetilde{f}$ is orientation preserving. $\widetilde{f}$ induces a homeomorphism $\mathcal{EG} \to \mathcal{EG}'$ between the embedded graphs $\mathcal{EG} = \cup_{i=1}^n a_i$ and $\mathcal{EG}' = \cup_{i=1}^n a_i'$. This induced homeomorphism is

$$e_i^- \mapsto \begin{cases} (e_i^+)' & \text{when } i \text{ is odd} \\ (e_i^-)' & \text{when } i \text{ is even} \end{cases}$$

$$e_i^+ \mapsto \begin{cases} (e_i^-)' & \text{when } i \text{ is odd} \\ (e_i^+)' & \text{when } i \text{ is even} \end{cases}$$

In particular, $\widetilde{f}(a_i) = a_i'$ when i is even and $\widetilde{f}(a_i) = (a_i')^{-1}$ when i is odd.

Now assume that n is even. By theorem 4.3.4, N_a (and N_a') is homeomorphic to either $S_{\frac{n-2}{2},4}$ or $S_{\frac{n}{2},2}$. Assuming that N_a and N_a' have the same topological type, define $f : \partial N_a \to \partial N_a'$ by

$$\partial N_a \ni Re_i^{\pm} \mapsto R(e_i^{\pm})' \in \partial N_a'$$
$$\partial N_a \ni Le_i^{\pm} \mapsto L(e_i^{\pm})' \in \partial N_a'$$

Since the boundary segments match and the intersection between the boundary segments are preserved, f extends to a homeomorphism $\widetilde{f} : N_a \to N_a'$. Choose the suitable orientation on N_a' so that $\widetilde{f}$ is an orientation preserving. $\widetilde{f}$ induces a homeomorphism $\mathcal{EG} \to \mathcal{EG}'$ between the graphs $\mathcal{EG} = \cup_{i=1}^{n} a_i$ and $\mathcal{EG}' = \cup_{i=1}^{n} a_i'$, given by:

$$\mathcal{EG} \ni e_i^{+} \mapsto (e_i^{+})' \in \mathcal{EG}'$$
$$\mathcal{EG} \ni e_i^{-} \mapsto (e_i^{-})' \in \mathcal{EG}'$$

In particular, $f(a_i) = a_i'$ for each $i = 1, \cdots, n$. $\square$

Example. We illustrate theorem 5.5.3 when $n = 3$, N_c is of type I, and N_c' is of type II. By theorem 4.3.4, N_c has three boundary cycles C_1, C_2, C_3 of lengths 6, 3, and 3. C_1, C_2, and C_3 are given by

$$\begin{tikzcd} Re_1^- \arrow[r] & Le_2^+ \arrow[r] & Re_3^- \arrow[d] \\ Le_3^+ \arrow[u] & Re_2^- \arrow[l] & Le_1^+ \arrow[l] \end{tikzcd}$$

$$\begin{tikzcd} Re_1^+ \arrow[r] & Re_2^+ \arrow[r] & Re_3^+ \arrow[ll, bend left] \end{tikzcd}$$

$$\begin{tikzcd} Le_1^- \arrow[r] & Le_2^- \arrow[r] & Le_3^- \arrow[ll, bend left] \end{tikzcd}$$

respectively. Figure 5.6 shows C_1 (in blue), C_2 (in red), and C_3 (in yellow). N_c' has three boundary cycles C_1', C_2', C_3' of lengths 6, 3, and 3 as well. These cycles have respective

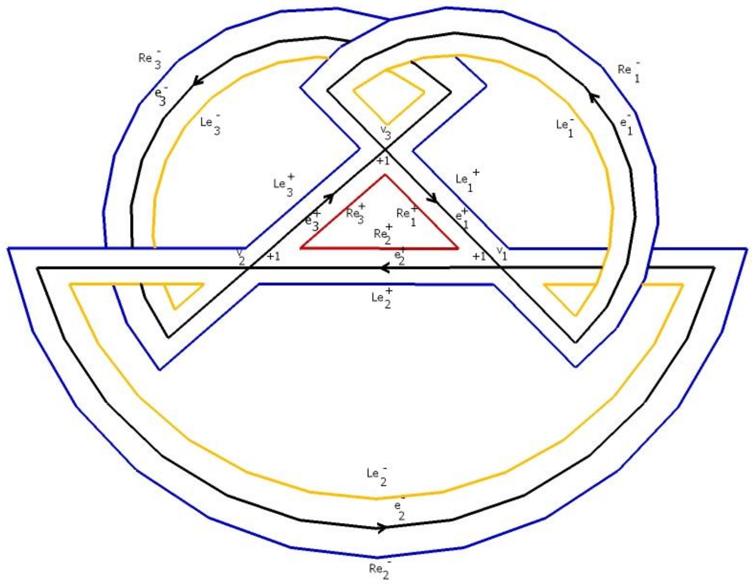


Figure 5.6: N_a is of type I.

colors blue, red, and yellow in figure 5.7. Moreover, C_1', C_2', and C_3' are given by

$$\begin{array}{ccccc} R(e_1^+)' & \longrightarrow & R(e_2^+)' & \longrightarrow & R(e_3^+)' \\ \uparrow & & & & \downarrow \\ L(e_3^-)' & \longleftarrow & L(e_2^-)' & \longleftarrow & L(e_1^-)' \end{array}$$

$$R(e_1^-)' \longrightarrow L(e_2^+)' \longrightarrow R(e_3^-)' \quad (R(e_3^-)' \to R(e_1^-)')$$

$$L(e_1^+)' \longrightarrow R(e_2^-)' \longrightarrow L(e_3^+)' \quad (L(e_3^+)' \to L(e_1^+)')$$

The homeomorphism $f : \partial N_a \to \partial N_a'$ defined by

$$\partial N_a \ni Re_1^- \mapsto R(e_1^+)' \in \partial N_a'$$
$$\partial N_a \ni Le_2^+ \mapsto R(e_2^+)' \in \partial N_a'$$
$$\partial N_a \ni Re_3^- \mapsto R(e_3^+)' \in \partial N_a'$$

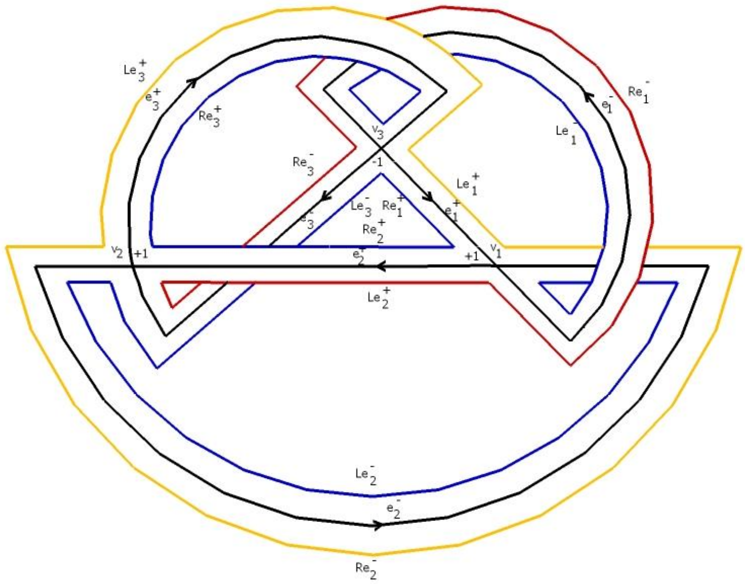


Figure 5.7: N'_a is of type II.

$$\partial N_a \ni Le_1^+ \mapsto L(e_1^-)' \in \partial N'_a$$
$$\partial N_a \ni Re_2^- \mapsto L(e_2^-)' \in \partial N'_a$$
$$\partial N_a \ni Le_3^+ \mapsto L(e_3^-)' \in \partial N'_a$$
$$\partial N_a \ni Le_1^- \mapsto L(e_1^+)' \in \partial N'_a$$
$$\partial N_a \ni Re_2^+ \mapsto L(e_2^+)' \in \partial N'_a$$
$$\partial N_a \ni Le_3^- \mapsto L(e_3^+)' \in \partial N'_a$$
$$\partial N_a \ni Re_1^+ \mapsto R(e_1^-)' \in \partial N'_a$$
$$\partial N_a \ni Le_2^- \mapsto R(e_2^-)' \in \partial N'_a$$
$$\partial N_a \ni Re_3^+ \mapsto R(e_3^-)' \in \partial N'_a$$

extends to a homeomorphism $\widetilde{f} : N_a \to N'_a$ such that $f(a_i) = a'_i$ (with no orientation). Therefore, $\widetilde{f}$ induces an isomorphism $G \cong G'$ between the subgroup $G < Mod(N_a)$ generated by the Dehn twists T_i along a_i, $i = 1, 2, 3$ and the subgroup $G' < Mod(N'_a)$ generated by the Dehn twists T'_i.

Theorem 5.5.4. *Consider a collection $\{a_1, \cdots, a_n\}$ of simple closed curves in some orientable surface S, and assume that the a_i have curve graph $\widetilde{A}_{n-1}$. Let N_a be a closed* *regular neighborhood of* $\cup_{i=1}^{n} a_i$*. Then,* N_a *is homeomorphic to the surface* $\widetilde{S}$ *constructed in section 4.3. By theorem 4.3.4,* $\widetilde{S}$ *has topological type* $S_{\frac{n-1}{2},3}$ *when* n *is odd, and either* $S_{\frac{n}{2},2}$ *or* $S_{\frac{n-2}{2},4}$ *when* n *is even. Assume that* N_a *is homeomorphic only to* $S_{\frac{n-1}{2},3}$ *or* $S_{\frac{n}{2},2}$*. If* T_i *represents the left Dehn twist along* a_i *and* G *denotes the subgroup of* $Mod(N_a)$ *generated by* $T_1, \cdots, T_n$*, then*

$$\begin{aligned} G &= \langle T_1, \cdots, T_n | \; T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \;\; mod(n) \; for \;\; i = 1, \cdots, n \\ &\quad T_i T_j = T_j T_i \; for \; |i-j| \neq 1, n-1 \rangle \end{aligned}$$

In particular, $\mathcal{A}(\widetilde{A}_{n-1}) \cong G$ *via the geometric homomorphism which maps the standard generator* σ_i *of* $\mathcal{A}(\widetilde{A}_{n-1})$ *to* T_i*.*

Proof. When n is odd, N_a is a well-defined surface, which is homeomorphic to $S_{\frac{n-1}{2},3}$, by theorem 4.3.4. Let $\{c_1, \cdots, c_n\}$ and $\{c'_1, \cdots, c'_n\}$ be arbitrary collections of simple closed curves in S, with curve graphs $\widetilde{A}_{n-1}$. A closed regular neighborhood N_c of $\cup_{i=1}^{n} c_i$ is homeomorphic to a closed regular neighborhood N'_c of $\cup_{i=1}^{n} c'_i$. By lemma 5.5.3, there is an orientation-preserving homeomorphism $f : N_c \to N'_c$ such that $f(c_i) = c'_i$. If $\tau_1, \cdots, \tau_n$ represent the left Dehn twists along $c_1, \cdots, c_n$ and $\tau'_1, \cdots, \tau'_n$ represent the left Dehn twists along $c'_1, \cdots, c'_n$, then f induces an isomorphism between the subgroup of $Mod(N_c)$ generated by the τ_i and that of $Mod(N'_c)$ generated by the τ'_i. As such, for any configuration of $c_1, \cdots, c_n$ with curve graph $\widetilde{A}_{n-1}$, the subgroup of $Mod(N_c)$ generated by the Dehn twists $\tau_1, \cdots, \tau_n$ is unique up to isomorphism. Due to this uniqueness, we may assume choose the closed curves $a_1, \cdots, a_n$ (in the hypothesis) as follows. Pick an arbitrary n-chain $b_1, \cdots, b_n$ of simple closed curves in S, then set $a_j = b_{j+1}$ for $j = 1, \cdots, n-1$ and $a_n = T_{b_1}^2 T_{b_2} \cdots T_{b_{n-1}}(b_n)$, where T_{b_i} is the left Dehn twist along b_i. The curves a_i, $i = 1, \cdots, n$ are illustrated in Figure 5.8. Note that since $a_j = b_{j+1}$, fact 1.3.2 implies $T_j = T_{b_{j+1}}$ (recall T_j is the left Dehn twist along a_j). Let T_n be the left Dehn twist along a_n. By theorem 5.5.1, the a_i have curve graph $\widetilde{A}_{n-1}$. Moreover, proposition 5.5.2 gives $\mathcal{A}(\widetilde{A}_{n-1}) \cong G$ via the geometric homomorphism $\sigma_i \mapsto T_i$.

When n is even, N_a is either homeomorphic to $S_{\frac{n}{2},2}$ or $S_{\frac{n-2}{2},4}$. By hypothesis, we

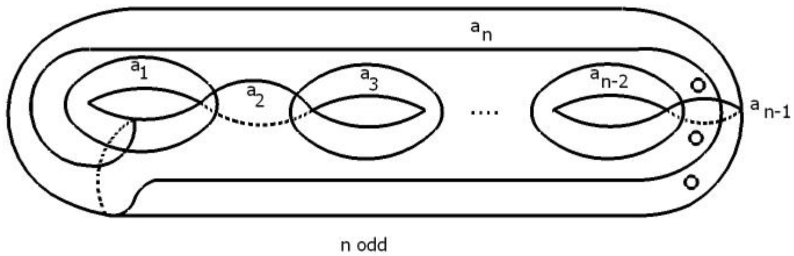


Figure 5.8: The collection $\{a_1, \cdots, a_n\}$ (n odd) has curve graph $\widetilde{A}_{n-1}$. The a_i are shown in the surface N_a which is homeomorphic to a closed regular neighborhood of $\cup_{i=1}^n a_i$. According to theorem 5.5.4, the subgroup G of $Mod(N_a)$ generated by the Dehn twists T_i along a_i is isomorphic to $\mathcal{A}(\widetilde{A}_{n-1})$.

only consider the case when N_a has topological type $S_{\frac{n}{2},2}$. Consider an arbitrary collection $\{c_1, \cdots, c_n\}$ of simple closed curves in S with curve graph $\widetilde{A}_{n-1}$. If a closed regular neighborhood N_c of $\cup_{i=1}^n c_i$ is homeomorphic to $S_{\frac{n}{2},2}$, it follows from lemma 5.5.3 that the subgroup of $Mod(N_c)$ generated by the left Dehn twists $\tau_1, \cdots, \tau_n$ along $c_1, \cdots, c_n$ is unique up to isomorphism. Hence, without loss of generality, the collection $a_1, \cdots, a_n$ in the hypothesis may be chosen by picking an arbitrary n-chain $b_1, \cdots, b_n$ of simple closed curves in S, then setting $a_j = b_{j+1}$ for $j = 1, \cdots, n-1$ and $a_n = T_{b_1}^2 T_{b_2} \cdots T_{b_{n-1}}(b_n)$, where T_{b_i} denotes the left Dehn twist along b_i. The curves a_i, $i = 1, \cdots, n$ are illustrated in figure 5.9. Let T_n be the left Dehn twist along a_n, and recall that N_a denotes a closed regular neighborhood of $\cup_{i=1}^n a_i$. By theorem 5.5.1, the a_i have curve graph $\widetilde{A}_{n-1}$. Moreover, it is easy to check that the embedded graph $\cup_{i=1}^n a_i$ is homeomorphic to $S_{\frac{n}{2},2}$. By proposition 5.5.2, the subgroup G of $Mod(N_a)$ generated by $T_1, \cdots, T_n$ has the desired presentation and is isomorphic to $\mathcal{A}(\widetilde{A}_{n-1})$ via the geometric homomorphism $\sigma_i \mapsto T_i$. □

We remark that when $\widetilde{S}$ is a subsurface of S such that no component of the closure of $S \backslash \widetilde{S}$ is an exterior cylinder or a disk with less than two punctures, it follows from corollary 1.5.3 that $i_* : Mod(\widetilde{S}) \to Mod(S)$ in injective. Hence, $T_1, \cdots, T_n$ generate $\mathcal{A}(\widetilde{A}_{n-1})$ in $Mod(S)$.

Finally, note that the case when n is even and $\widetilde{S}$ is of type I is excluded from theorem 5.5.4. This is simply because we are not able to find a geometric embedding from CB_n

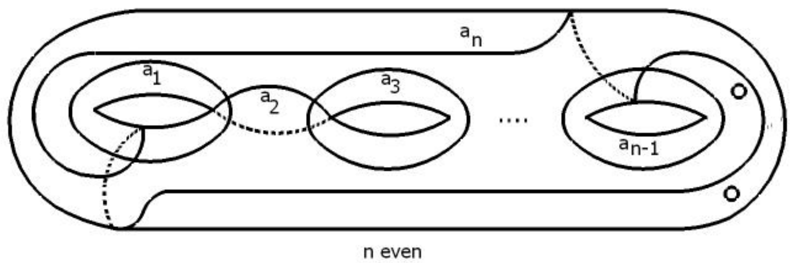


Figure 5.9: The collection $\{a_1, \cdots, a_n\}$ (n even) has curve graph $\widetilde{A}_{n-1}$. The a_i are shown in the surface which is homeomorphic to a closed regular neighborhood of $\cup_{i=1}^{n} a_i$. By theorem 5.5.4, $G \cong \mathcal{A}(\widetilde{A}_{n-1})$.

into $Mod(S)$ so that N_a has type I. Nevertheless, we do make the following conjecture:

Conjecture 5.5.5. *Theorem 5.5.4 holds when n is even and $\widetilde{S}$ is of type I.*

CHAPTER 6

ARTIN RELATIONS IN MOD(S)

In this chapter, we first define the Artin relation of length l. Then, we find elements in $Mod(S)$ satisfying such a relation for every positive integer l. In certain cases, the elements we find generate an Artin group of type $I_2(l)$. These cases provide embeddings of $\mathcal{A}(I_2(l))$ into $Mod(S)$.

In section 6.1, we find elements x and y in $Mod(S)$ satisfying Artin relations of every even length $l \geq 8$ (see theorem 6.1.1). In section 6.2, we find Artin relations of every odd length $l \geq 3$ in $Mod(S)$ (see theorem 6.2.1). Finally, in section 6.3, we use foldings to produce more Artin relations in $Mod(S)$. Theorem 6.3.1 produces Artin relations of every length $l \geq 3$, and theorem 6.3.2 produces Artin relations of every even length $l \geq 6$. In theorems 6.3.1 and 6.3.2, x and y generate the Artin group $\mathcal{A}(I_2(l))$ in $Mod(S)$.

Definition 6.0.6. *If $l \geq 2$ is an integer, we say that elements a and b in a group G satisfy the Artin relation of length l (or the l-Artin relation) if $prod(a,b;l) = prod(b,a;l)$, where*

$$prod(a,b;l) = \underbrace{aba\cdots}_{l}$$

6.1 Artin relations of even length

In this section we find Artin relations of even lengths. If n is a positive integer multiple of $2k+4$, $k \geq 2$, we find elements x and y in the mapping class group of some appropriate

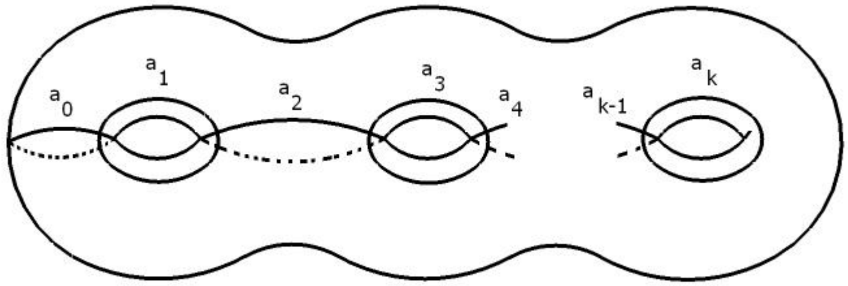


Figure 6.1: The curves $a_0, \cdots, a_k$ form a chain of length $k+1$. If $x = T_0$ and $y = T_1 \cdots T_k$, then $prod(x, y; 2k + 4) = prod(y, x; 2k + 4)$.

orientable surface, so that $prod(x, y; n) = prod(y, x; n)$. By appropriate orientable surface, we mean one with large enough genus to accommodate a chain of $k + 1$ curves.

Theorem 6.1.1. *Let $k \geq 2$ be an integer. Suppose $a_0, a_1, \cdots, a_k$ form a chain of simple closed curves in an orientable surface S. If $x = T_0$ and $y = T_1 \cdots T_k$, then $prod(x, y; l) = prod(y, x; l) \Leftrightarrow l \equiv 0(mod(2k + 4))$.*

Proof. In the proof, we shall only cancel on the left. Right cancellations are intentionally ignored. This simplifies the proof, as it allows us to start a new computation by using the result from the previous one. Throughout the proof, we shall use facts 1.3.7 and 1.3.8.

$$\begin{aligned} xy = yx \Leftrightarrow T_0T_1 \cdots T_k &= T_1T_2 \cdots T_kT_0 \\ \Leftrightarrow T_0T_1 \cdots T_k &= T_1T_0T_2 \cdots T_k \end{aligned}$$

Since $i(a_0, a_1) = 1$, $T_0T_1 \neq T_1T_0$. As such, the last equality on the right hand side (RHS) does not hold.

We now describe a method that will be used later without further explicit mention. The equation $xy = yx$ above does not hold. However, the algebraic manipulations for the equivalence of $xy = yx$ with the last equation above do hold. Next, we multiply these equations by x and y on the right accordingly. The computations above imply:

$$\begin{aligned}
xyx = yxy \Leftrightarrow (T_0T_1\cdots T_k)T_0 &= T_1T_0T_2\cdots T_k(T_1\cdots T_k) \\
\Leftrightarrow (T_0T_1T_0)T_2\cdots T_k &= T_1T_0T_2T_1T_3T_2T_4T_3\cdots T_iT_{i-1}T_{i+1}T_i\cdots \\
&\quad T_{k-1}T_{k-2}(T_kT_{k-1}T_k) \\
\Leftrightarrow T_1T_0T_1T_2\cdots T_k &= T_1T_0T_2T_1T_3T_2T_4T_3\cdots T_iT_{i-1}T_{i+1}T_i\cdots \\
&\quad (T_{k-1}T_{k-2}T_{k-1})T_kT_{k-1} \\
\Leftrightarrow T_1T_0T_1T_2\cdots T_k &= T_1T_0T_1T_2T_1T_3T_2T_4T_3\cdots T_iT_{i-1}T_{i+1}T_i\cdots \\
&\quad T_{k-3}T_{k-1}T_{k-2}T_kT_{k-1} \\
\Leftrightarrow T_1T_0T_1T_2\cdots T_k &= T_1T_0T_1T_2T_3T_4\cdots T_kT_1T_2T_3\cdots T_{k-2}T_{k-1} \\
\Leftrightarrow 1 &= T_1\cdots T_{k-1}
\end{aligned}$$

Since $T_1\cdots T_{k-1}(a_2) = a_3$ or $T_1(a_2)$, the last equation of RHS does not hold.

Using the above equivalences of $xyx = yxy$, and only left cancellation, we do the following for $(xy)^2 = (yx)^2$.

$$\begin{aligned}
(xy)^2 = (yx)^2 \Leftrightarrow T_1\cdots T_{k-1}T_k &= T_1\cdots T_{k-1}T_0 \\
\Leftrightarrow T_k &= T_0
\end{aligned}$$

Since $k \geq 2$ by assumption, the last equality of RHS is obviously not true.

$$\begin{aligned}
(xy)^2x = (yx)^2y \Leftrightarrow T_kT_0 &= T_0T_1\cdots T_k \\
\Leftrightarrow T_0T_k &= T_0T_1\cdots T_k \\
\Leftrightarrow T_k &= T_1\cdots T_k
\end{aligned}$$

Since $1 \neq T_1\cdots T_{k-1}$ (see above), the last equality of RHS does not hold.

$$(xy)^3 = (yx)^3 \Leftrightarrow T_kT_1\cdots T_{k-2}T_{k-1}T_k = T_1\cdots T_kT_0$$

$$\begin{aligned} \Leftrightarrow T_1 \cdots T_{k-2}(T_k T_{k-1} T_k) &= T_1 \cdots T_k T_0 \\ \Leftrightarrow T_1 \cdots T_{k-2} T_{k-1} T_k T_{k-1} &= T_1 \cdots T_k T_0 \\ \Leftrightarrow T_{k-1} &= T_0 \end{aligned}$$

Since $k \geq 2$ by assumption, $T_{k-1} \neq T_0$.

$$(xy)^3 x = (yx)^3 y \Leftrightarrow T_{k-1} T_0 = T_0 T_1 \cdots T_k$$

If $k = 2$, then $(xy)^3 x = (yx)^3 y \Leftrightarrow T_1 T_0 = T_0 T_1 T_2$. Otherwise, $k \geq 3$. In this case,

$$\begin{aligned} (xy)^3 x = (yx)^3 y \Leftrightarrow T_{k-1} T_0 &= T_0 T_1 \cdots T_k \\ \Leftrightarrow T_0 T_{k-1} &= T_0 T_1 \cdots T_k \\ \Leftrightarrow T_{k-1} &= T_1 \cdots T_k \end{aligned}$$

Since $i(a_{k-1}, a_k) = 1$, $T_{k-1} T_k(a_{k-1}) = a_k$. Hence, $T_1 \cdots T_k(a_{k-1}) = a_k \neq a_{k-1}$ implies that $T_{k-1} \neq T_1 \cdots T_k$ when $k > 2$. Moreover, $T_1 T_0(a_1) = a_0 \neq a_2 = T_0 T_1 T_2(a_1)$ implies $T_1 T_0 \neq T_0 T_1 T_2$.

When $k = 2$, it follows from the previous equivalence that $(xy)^4 = (yx)^4 \Leftrightarrow T_1 T_0 T_1 T_2 = T_0 T_1 T_2 T_0 \Leftrightarrow T_0 T_1 T_0 T_2 = T_0 T_1 T_0 T_2$, which is obviously true. This shows that $prod(x, y; 2k+4) = prod(y, x; 2k+4)$ for $k = 2$ (in this case $x = T_0$ and $y = T_1 T_2$). When $k > 2$, we have:

$$\begin{aligned} (xy)^4 = (yx)^4 \Leftrightarrow T_{k-1} T_1 \cdots T_{k-2} T_{k-1} T_k &= T_1 \cdots T_k T_0 \\ \Leftrightarrow T_1 \cdots T_{k-3}(T_{k-1} T_{k-2} T_{k-1}) T_k &= T_1 \cdots T_k T_0 \\ \Leftrightarrow T_1 \cdots T_{k-3} T_{k-2} T_{k-1} T_{k-2} T_k &= T_1 \cdots T_k T_0 \\ \Leftrightarrow T_1 \cdots T_{k-3} T_{k-2} T_{k-1} T_k T_{k-2} &= T_1 \cdots T_k T_0 \\ \Leftrightarrow T_{k-2} &= T_0 \end{aligned}$$

which is obviously false because $k > 2$.

To prove the theorem in general, we make the following claims :

Claim 6.1.2. *Let k and i be positive integers such that $k \geq 2$ and $3 \leq i \leq k+1$. Then, for all i, $prod(x, y; 2i-1) = prod(y, x; 2i-1) \Leftrightarrow T_{k-i+3} = T_1 \cdots T_k$.*

Claim 6.1.3. *Let k and i be positive integers such that $k \geq 2$ and $3 \leq i \leq k+1$. Then, for all i, $prod(x, y; 2i) = prod(y, x; 2i) \Leftrightarrow T_{k-i+2} = T_0$.*

To prove claim 6.1.2, we proceed by induction on i. The base case, $i = 3$, has been proven above. Assume, by induction, that claim 6.1.2 holds for some $i \in \{3, \cdots, k\}$. We would like to show claim 6.1.2 holds for $i+1$. That is, we need to prove:

$$\begin{aligned} prod(x, y; 2i+1) = prod(y, x; 2i+1) \Leftrightarrow T_{k-(i+1)+3} &= T_1 \cdots T_k \\ \Leftrightarrow T_{k-i+2} &= T_1 \cdots T_k \end{aligned}$$

Assuming claim 6.1.2 for i implies that $prod(x, y; 2i) = prod(y, x; 2i)$

$$\begin{aligned} \Leftrightarrow T_{k-i+3}T_1 \cdots T_{k-i+2}T_{k-i+3} \cdots T_k &= T_1 \cdots T_{k-i+2} \cdots T_k T_0 \\ \Leftrightarrow T_1 \cdots (T_{k-i+3}T_{k-i+2}T_{k-i+3}) \cdots T_k &= T_1 \cdots T_{k-i+2} \cdots T_k T_0 \\ \Leftrightarrow T_{k-i+2}T_{k-i+3}T_{k-i+2}T_{k-i+4} \cdots T_k &= T_{k-i+2} \cdots T_k T_0 \\ \Leftrightarrow T_{k-i+2} \cdots T_k T_{k-i+2} &= T_{k-i+2} \cdots T_k T_0 \\ T_{k-i+2} &= T_0 \end{aligned}$$

To justify the above calculation, note that the i under consideration belongs to $\{3, \cdots, k\}$. Since $k \geq 2$, it follows that $k - i + 3 \in \{3, \cdots, k\}$. As such, $[T_{k-i+3}, T_1] = 1$. In particular, this shows that claim 6.1.2 for some positive integer i, $3 \leq i \leq k+1$ and $k \geq 2$, implies claim 6.1.3 for that i. Given the equivalence $prod(x, y; 2i) = prod(y, x; 2i) \Leftrightarrow T_{k-i+2} = T_0$, then

$$\begin{aligned} prod(x,y;2i+1) = prod(y,x;2i+1) \Leftrightarrow T_{k-i+2}T_0 &= T_0T_1\cdots T_k \\ \Leftrightarrow T_0T_{k-i+2} &= T_0T_1\cdots T_k \\ \Leftrightarrow T_{k-i+2} &= T_1\cdots T_k \end{aligned}$$

The above calculation is justified because $k-i+2 \in \{2,\cdots,k-1\}$, and so $[T_{k-i+2}, T_0] = 1$. This concludes the proof of claim 6.1.2.

By the above remark, the proof of claim 6.1.3 follows immediately from claim 6.1.2 and its proof.

Now we prove theorem 6.1.1. Assume $l \equiv 0(mod(2k+4))$. As $prod(x,y;2k+4) = prod(y,x;2k+4) \Rightarrow prod(x,y;q(2k+4)) = prod(y,x;q(2k+4))$ for all positive integers q, it suffices to show $prod(x,y;2k+4) = prod(y,x;2k+4)$. By claim 6.1.3, we have:

$$\begin{aligned} prod(x,y;2k+2) &= prod(y,x;2k+2) \Leftrightarrow \\ T_{k-(k+1)+2} &= T_0 \Leftrightarrow \\ T_1 &= T_0 \end{aligned}$$

which is not true. Given this, then

$$prod(x,y;2k+3) = prod(y,x;2k+3) \Leftrightarrow T_1T_0 = T_0T_1\cdots T_k$$

Since $T_1T_0(a_1) = a_0 \neq a_2 = T_0T_1\cdots T_k(a_1)$, $T_1T_0 \neq T_0T_1\cdots T_k$. Finally,

$$\begin{aligned} prod(x,y;2k+4) = prod(y,x;2k+4) \Leftrightarrow (T_1T_0T_1)\cdots T_k &= T_0T_1\cdots T_kT_0 \\ \Leftrightarrow T_0T_1T_0\cdots T_k &= T_0T_1T_0\cdots T_k \end{aligned}$$

which is true.

Conversely, assume l is not a multiple of $2k+4$. Then, $prod(x,y;l) = prod(y,x;l)$ if and only if $prod(x,y;r) = prod(y,x;r)$ for some $r \in \{1,\cdots,2k+3\}$, where $l \equiv r(mod(2k+4))$.

If $r = 2k+3$, it was shown above that $prod(x,y;r) \neq prod(y,x;r)$. If $r < 2k+3$ is odd; say $r = 2s-1$ for some positive integer s, then $prod(x,y;r) = prod(y,x;r) \Leftrightarrow T_{k-s+3} = T_1 \cdots T_k$ by claim 6.1.2. But then $T_{k-s+3}(a_{k-s+3}) = a_{k-s+3}$ while

$$\begin{aligned} T_1 \cdots T_k(a_{k-s+3}) &= T_1 \cdots T_{k-s+2} T_{k-s+3} T_{k-s+4}(a_{k-s+3}) \\ &= T_1 \cdots T_{k-s+2}(a_{k-s+4}) \\ &= a_{k-s+4} \neq a_{k-s+3} \end{aligned}$$

If r is even; say $r = 2s$, then by claim 6.1.3, $prod(x,y;r) = prod(y,x;r) \Leftrightarrow T_{k-s+2} = T_0 \Leftrightarrow k-s+2 = 0 \Leftrightarrow l = 2k+4$. □

Conjecture 6.1.4. *Let $a_0, a_1, \cdots, a_k$ and T_i, $i \in \{0,\cdots,k\}$ be as in theorem 6.1.1. Let $x = T_0$ and $y = T_{\sigma(1)} T_{\sigma(2)} \cdots T_{\sigma(k)}$, where $\sigma \in \Sigma_k$. Then $prod(x,y;n) = prod(y,x;n) \Leftrightarrow n \equiv 0(mod(2k+4))$.*

The conjecture holds when $k = 2, 3$, and 4. This has been proven by brute force calculations. For $k = 4$, the are six permutations (including the one of theorem 6.1.1), and $2k+4 = 10$.

6.2 Artin relations of odd length

In this section, we find Artin relations of every odd length in the mapping class group. More precisely, given an odd positive integer n, we find elements in the mapping class group of some appropriate orientable surface satisfying the Artin relation of length n. In a way, the relations we discover are generalizations of the famous braid relation in proposition 1.3.8, to all odd lengths.

Notation. In theorem 6.2.1 below, we shall change the notation of a Dehn twist in order

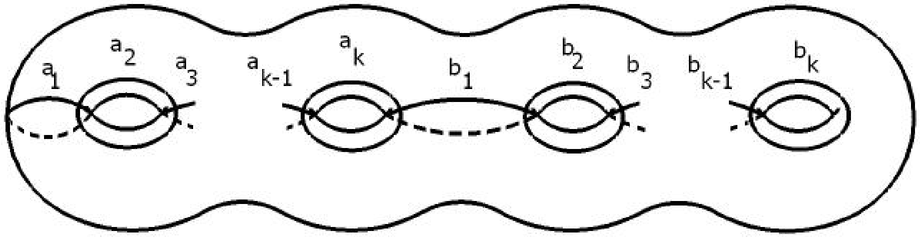


Figure 6.2: The curves $a_1, \cdots, a_k, b_1, \cdots, b_k$ form a chain of length $2k$, $k \geq 1$. If $x = A_1 \cdots A_k$ and $y = B_1 \cdots B_k$, then $prod(x, y; 2k+1) = prod(y, x; 2k+1)$.

to make it easier for the reader to follow the proof. Instead of T_i, we shall denote Dehn twists by A_i and B_i. A_i and B_i represent Dehn twists along curves a_i and b_i respectively.

Theorem 6.2.1. *Let $k \geq 1$ be an integer, and suppose that the simple closed curves $a_1, \cdots, a_k, b_1, \cdots, b_k$ form a 2k-chain in an orientable surface S. If $x = A_1 \cdots A_k$ and $y = B_1 \cdots B_k$, then $prod(x, y; l) = prod(y, x; l) \Leftrightarrow l \equiv 0(mod(2k+1))$.*

Proof. In this proof, we again cancel only from the left. This allows us to pick up where we left when studying the next Artin relation. Right cancellations are intentionally ignored. When $k = 1$, there are two curves a_1 and b_1 with $i(a_1, b_1) = 1$. Hence, $x = A_1$ and $y = B_1$, and by fact 1.3.8, $xyx = yxy$. Consequently, $prod(x, y; l) = prod(y, x; l)$ for all positive integers l that are multiples of 3. Conversely, suppose $l \not\equiv 0(mod3)$. If $prod(x, y; l) = prod(y, x; l)$, then $prod(x, y; r) = prod(y, x; r)$ for some $r \in \{1, 2\}$. Since $i(a_1, b_1) = 1$, $x \neq y$. Moreover, if $xy = yx$, it follows from $xyx = yxy$ that $x = y$, which is a contradiction. This proves the theorem for $k = 1$. So, we henceforth assume that $k \geq 2$.

We remark that when B_{k-2} or B_{k-3} are included in any of the computations below, it should be assumed that $k > 3$. The inclusion of such terms in the rather complicated calculations is intended to help the reader follow the proof. Although not included, the calculations for $k \in \{2, 3\}$ follow along the same lines of the ones shown (for $k > 3$). In fact, the cases $k \in \{2, 3\}$ are much easier because there are less terms involved.

$$xy = yx \Leftrightarrow A_1 \cdots A_{k-1} A_k B_1 \cdots B_k = B_1 \cdots B_k A_1 \cdots A_{k-1} A_k$$

$$\begin{aligned}\Leftrightarrow A_1 \cdots A_{k-1} A_k B_1 \cdots B_k &= A_1 \cdots A_{k-1} B_1 \cdots B_k A_k \\ \Leftrightarrow A_k B_1 \cdots B_k &= B_1 A_k B_2 \cdots B_k\end{aligned}$$

Since $[A_k, B_1] \neq 1$, the last equality of RHS does not hold.

$$(xy)x = (yx)y \Leftrightarrow A_k B_1 \cdots B_k A_1 \cdots A_k = B_1 A_k B_2 \cdots B_k B_1 \cdots B_{k-1} B_k$$

Set $\delta_1 = A_k B_1 \cdots B_k A_1 \cdots A_k$. Then

$$\begin{aligned}(xy)x = (yx)y \Leftrightarrow \delta_1 &= B_1 A_k B_2 \cdots B_{k-1} B_1 \cdots B_{k-2} B_k B_{k-1} B_k \\ &= B_1 A_k B_2 \cdots B_{k-2} B_1 \cdots B_{k-1} B_{k-2} B_k B_{k-1} B_k \\ &\vdots \\ &= B_1 A_k B_2 B_1 B_3 B_2 \cdots B_i B_{i-1} B_{i+1} B_i \cdots \\ &\quad B_{k-2} B_{k-3} B_{k-1} B_{k-2} (B_k B_{k-1} B_k) \\ &= B_1 A_k B_2 B_1 B_3 B_2 \cdots B_i B_{i-1} B_{i+1} B_i \cdots \\ &\quad B_{k-2} B_{k-3} (B_{k-1} B_{k-2} B_{k-1}) B_k B_{k-1} \\ &= B_1 A_k B_2 B_1 B_3 B_2 \cdots B_i B_{i-1} B_{i+1} B_i \cdots \\ &\quad (B_{k-2} B_{k-3} B_{k-2}) B_{k-1} B_{k-2} B_k B_{k-1} \\ &\vdots \\ &= (B_1 A_k B_1) B_2 B_1 B_3 B_2 B_4 B_3 \cdots B_i B_{i-1} B_{i+1} B_i \cdots \\ &\quad B_{k-3} B_{k-2} B_{k-3} B_{k-1} B_{k-2} B_k B_{k-1} \\ &= A_k B_1 A_k B_2 B_1 B_3 B_2 B_4 B_3 \cdots B_i B_{i-1} B_{i+1} B_i \cdots \\ &\quad B_{k-3} B_{k-2} B_{k-3} B_{k-1} B_{k-2} B_k B_{k-1} \\ &\vdots \\ &= A_k B_1 B_2 B_3 \cdots B_k A_k B_1 B_2 B_3 \cdots B_{k-1}\end{aligned}$$

In order to get the last expression above, we shifted the second A_k to the right as much as possible, and the B_i's to the left as much as possible. Similar shifts occur in future computations.

$$\begin{aligned}(xy)x = (yx)y \Leftrightarrow A_kB_1\cdots B_kA_1\cdots A_k &= A_kB_1\cdots B_kA_kB_1\cdots B_{k-1}\\ \Leftrightarrow A_1\cdots A_k &= A_kB_1\cdots B_{k-1}\end{aligned}$$

Acting with the products $A_1\cdots A_k$ and $A_kB_1\cdots B_{k-1}$ on a_1 yields distinct curves. Consequently, the two products are distinct.

$$\begin{aligned}(xy)^2 = (yx)^2 \Leftrightarrow A_1\cdots A_kB_1\cdots B_k &= A_kB_1\cdots B_{k-1}A_1\cdots A_{k-2}A_{k-1}A_k\\ \Leftrightarrow A_1\cdots A_{k-2}A_{k-1}A_kB_1\cdots B_k &= A_1\cdots A_{k-2}A_kB_1\cdots B_{k-1}A_{k-1}A_k\\ \Leftrightarrow A_{k-1}A_kB_1\cdots B_k &= A_kA_{k-1}B_1A_kB_2\cdots B_{k-1}\end{aligned}$$

Since $A_{k-1}A_kB_1\cdots B_k(a_{k-1}) = a_k \neq b_1 = A_kA_{k-1}B_1A_kB_2\cdots B_{k-1}(a_{k-1})$, the two expressions are distinct.

$$\begin{aligned}(xy)^2x &= (yx)^2y \Leftrightarrow\\ A_{k-1}A_kB_1\cdots B_kA_1\cdots A_k &= A_kA_{k-1}B_1A_kB_2\cdots B_{k-1}B_1\cdots B_k\end{aligned}$$

Set $\delta_2 = A_{k-1}A_kB_1\cdots B_kA_1\cdots A_k$. Then $(xy)^2x = (yx)^2y \Leftrightarrow$

$$\begin{aligned}\delta_2 &= A_kA_{k-1}B_1A_kB_2\cdots B_{k-2}B_1\cdots B_{k-1}B_{k-2}B_{k-1}B_k\\ &= A_kA_{k-1}B_1A_kB_2\cdots B_{k-3}B_1\cdots B_{k-2}B_{k-3}B_{k-1}B_{k-2}B_{k-1}B_k\\ &\vdots\\ &= A_kA_{k-1}B_1A_kB_2B_1B_3B_2\cdots B_iB_{i-1}B_{i+1}B_i\cdots\\ &\quad B_{k-2}B_{k-3}(B_{k-1}B_{k-2}B_{k-1})B_k\\ &= A_kA_{k-1}B_1A_kB_2B_1B_3B_2\cdots B_iB_{i-1}B_{i+1}B_i\cdots\\ &\quad (B_{k-2}B_{k-3}B_{k-2})B_{k-1}B_{k-2}B_k\end{aligned}$$

$$
\begin{aligned}
&= A_kA_{k-1}B_1A_kB_2B_1B_3B_2\cdots B_iB_{i-1}B_{i+1}B_i\cdots \\
&\quad B_{k-3}B_{k-2}B_{k-3}B_{k-1}B_{k-2}B_k \\
&\vdots \\
&= (A_kA_{k-1}A_k)B_1A_kB_2B_1B_3B_2\cdots B_iB_{i-1}B_{i+1}B_i\cdots \\
&\quad B_{k-3}B_{k-2}B_{k-3}B_{k-1}B_{k-2}B_k \\
&= A_{k-1}A_kA_{k-1}B_1A_kB_2B_1B_3B_2\cdots B_iB_{i-1}B_{i+1}B_i\cdots \\
&\quad B_{k-3}B_{k-2}B_{k-3}B_{k-1}B_{k-2}B_k \\
&= A_{k-1}A_kB_1\cdots B_kA_{k-1}A_kB_1\cdots B_{k-2}
\end{aligned}
$$

$$
\begin{aligned}
(xy)^2x &= (yx)^2y \\
\Leftrightarrow A_{k-1}A_kB_1\cdots B_kA_1\cdots A_k &= A_{k-1}A_kB_1\cdots B_kA_{k-1}A_kB_1\cdots B_{k-2} \\
\Leftrightarrow A_1\cdots A_k &= A_{k-1}A_kB_1\cdots B_{k-2}
\end{aligned}
$$

Since $A_1\cdots A_k(a_1) = a_2 \neq a_1 = A_{k-1}A_kB_1\cdots B_{k-2}(a_1)$, the two expressions are different. By continuing in the same fashion, one can see that there are seemingly visible patterns governing Artin equalities. We make the following claims:

Claim 6.2.2. *Let k and m be a positive integers such that $k \geq 2$ and $m < k$. Then, for all m, $(xy)^m = (yx)^m \Leftrightarrow$*

$$
\begin{aligned}
A_{k-m+1}\cdots A_kB_1\cdots B_k &\overset{(1)}{=} A_{k-m+2}A_{k-m+1}A_{k-m+3}A_{k-m+2}A_{k-m+4}A_{k-m+3} \\
&\quad A_{k-m+5}A_{k-m+4}\cdots A_{k-1}B_1A_kB_2\cdots B_{k-m+1}
\end{aligned}
$$

Claim 6.2.3. *Let k and m be a positive integers such that $k \geq 2$ and $m \leq k$. Then, for all m, $(xy)^mx = (yx)^my \Leftrightarrow A_1\cdots A_k \overset{(2)}{=} A_{k-m+1}\cdots A_kB_1\cdots B_{k-m}$*

In the claims above, k represents the number of A_i's in x (k is also equal to the number of B_i's in y). If l represents the lengths of the Artin relations considered in claims 6.2.2

and 6.2.3, then

$$m = \begin{cases} l/2 & \text{when } l \text{ is even} \\ \frac{l-1}{2} & \text{when } l \text{ is odd} \end{cases}$$

Proof of claims 6.2.2 and 6.2.3 - We proceed by induction on m. Suppose $(xy)^m = (yx)^m \Leftrightarrow$ (1) holds. First, we prove $(xy)^m x = (yx)^m y \Leftrightarrow$ (2) is true, then we show $(xy)^{m+1} = (yx)^{m+1} \Leftrightarrow$ (1) with m replaced with $m+1$. Fix an arbitrary m with $1 \leq m \leq k-2$, and assume claim 6.2.2 is true for that m.

$$\begin{aligned} (xy)^m x &= (yx)^m y \Leftrightarrow \\ A_{k-m+1} \cdots A_k B_1 \cdots B_k A_1 \cdots A_k &= A_{k-m+2} A_{k-m+1} A_{k-m+3} A_{k-m+2} A_{k-m+4} \\ &\quad A_{k-m+3} A_{k-m+5} A_{k-m+4} \cdots A_{k-1} B_1 A_k B_2 \\ &\quad \cdots B_{k-m+1} B_1 \cdots B_k \end{aligned}$$

Set $\delta_3 = A_{k-m+1} \cdots A_k B_1 \cdots B_k A_1 \cdots A_k$. Also, set $s = k - m$, and note that $s > 0$ since $m < k$. Then $(xy)^m x = (yx)^m y \Leftrightarrow$

$$\begin{aligned} \delta_3 &= A_{s+2} A_{s+1} A_{s+3} A_{s+2} A_{s+4} A_{s+3} A_{s+5} A_{s+4} \cdots A_{k-1} B_1 A_k B_2 \cdots \\ &\quad B_s B_1 \cdots B_{s-1} B_{s+1} B_s B_{s+1} \cdots B_k \\ &= A_{s+2} A_{s+1} A_{s+3} A_{s+2} A_{s+4} A_{s+3} A_{s+5} A_{s+4} \cdots A_{k-1} B_1 A_k B_2 \cdots \\ &\quad B_{s-1} B_1 \cdots B_{s-2} B_s B_{s-1} B_{s+1} B_s B_{s+1} \cdots B_k \\ &\vdots \\ &= A_{s+2} A_{s+1} A_{s+3} A_{s+2} A_{s+4} A_{s+3} A_{s+5} A_{s+4} \cdots A_{k-1} B_1 A_k B_2 B_1 B_3 B_2 B_4 B_3 \\ &\quad B_5 B_4 \cdots B_s B_{s-1} (B_{s+1} B_s B_{s+1}) B_{s+2} B_{s+3} \cdots B_k \\ &= A_{s+2} A_{s+1} A_{s+3} A_{s+2} A_{s+4} A_{s+3} A_{s+5} A_{s+4} \cdots A_{k-1} B_1 A_k B_2 B_1 B_3 B_2 B_4 B_3 \\ &\quad B_5 B_4 \cdots (B_s B_{s-1} B_s) B_{s+1} B_s B_{s+2} B_{s+3} \cdots B_k \end{aligned}$$

$$\begin{aligned}
&= A_{s+2}A_{s+1}A_{s+3}A_{s+2}A_{s+4}A_{s+3}A_{s+5}A_{s+4}\cdots A_{k-1}B_1A_kB_2B_1B_3B_2B_4B_3 \\
&\quad B_5B_4\cdots(B_{s-1}B_{s-2}B_{s-1})B_sB_{s-1}B_{s+1}B_sB_{s+2}B_{s+3}\cdots B_k \\
&\vdots \\
&= A_{s+2}A_{s+1}A_{s+3}A_{s+2}A_{s+4}A_{s+3}A_{s+5}A_{s+4}\cdots A_{k-1}B_1A_k(B_2B_1B_2)B_3B_2 \\
&\quad B_4B_3B_5B_4\cdots B_sB_{s-1}B_{s+1}B_sB_{s+2}B_{s+3}\cdots B_k \\
&= A_{s+2}A_{s+1}A_{s+3}A_{s+2}A_{s+4}A_{s+3}A_{s+5}A_{s+4}\cdots A_{k-1}(B_1A_kB_1)B_2B_1B_3B_2 \\
&\quad B_4B_3B_5B_4\cdots B_sB_{s-1}B_{s+1}B_sB_{s+2}B_{s+3}\cdots B_k \\
&= A_{s+2}A_{s+1}A_{s+3}A_{s+2}A_{s+4}A_{s+3}A_{s+5}A_{s+4}\cdots A_{k-1}A_{k-2}(A_kA_{k-1}A_k) \\
&\quad B_1A_kB_2B_1B_3B_2B_4B_3B_5B_4\cdots B_sB_{s-1}B_{s+1}B_sB_{s+2}B_{s+3}\cdots B_k \\
&\vdots \\
&= (A_{s+2}A_{s+1}A_{s+2})A_{s+3}A_{s+2}A_{s+4}A_{s+3}A_{s+5}A_{s+4}A_{s+6}A_{s+5}\cdots A_{k-1}A_{k-2} \\
&\quad A_kA_{k-1}B_1A_kB_2B_1B_3B_2B_4B_3B_5B_4\cdots B_sB_{s-1}B_{s+1}B_sB_{s+2}B_{s+3}\cdots B_k \\
&= A_{s+1}A_{s+2}A_{s+1}A_{s+3}A_{s+2}A_{s+4}A_{s+3}A_{s+5}A_{s+4}A_{s+6}A_{s+5}\cdots A_{k-1}A_{k-2}A_k \\
&\quad A_{k-1}B_1A_kB_2B_1B_3B_2B_4B_3B_5B_4\cdots B_sB_{s-1}B_{s+1}B_sB_{s+2}B_{s+3}\cdots B_k \\
&= A_{s+1}A_{s+2}A_{s+3}\cdots A_kB_1\cdots B_kA_{s+1}A_{s+2}A_{s+3}\cdots A_kB_1\cdots B_s
\end{aligned}$$

Thus, $(xy)^m x = (yx)^m y \Leftrightarrow$

$$\begin{aligned}
A_{s+1}A_{s+2}A_{s+3}\cdots A_kB_1\cdots B_kA_1\cdots A_k &= A_{s+1}A_{s+2}A_{s+3}\cdots A_kB_1\cdots B_k \\
\Leftrightarrow A_1\cdots A_k &= A_{s+1}A_{s+2}A_{s+3}\cdots A_kB_1\cdots B_s \\
\Leftrightarrow A_1\cdots A_k &= A_{k-m+1}A_{k-m+2}A_{k-m+3}\cdots A_k \\
&\quad \cdots B_1\cdots B_{k-m}
\end{aligned}$$

Note that this is claim 6.2.3 for m. We now show that claim 6.2.3 for m implies claim 6.2.2

for $m+1$. Again, we set $s = k - m$.

$$\begin{aligned}
(xy)^{m+1} &= (yx)^{m+1} \\
\Leftrightarrow A_1 \cdots A_k B_1 \cdots B_k &= A_{s+1} A_{s+2} A_{s+3} \cdots A_k B_1 \cdots B_s A_1 \cdots A_k \\
\Leftrightarrow A_1 \cdots A_{s-1} A_s \cdots A_k B_1 \cdots B_k &= A_1 \cdots A_{s-1} A_{s+1} A_{s+2} A_{s+3} \cdots A_k B_1 \cdots \\
& \quad B_s A_s A_{s+1} A_{s+2} \cdots A_k \\
\Leftrightarrow A_s \cdots A_k B_1 \cdots B_k &= A_{s+1} A_{s+2} A_{s+3} \cdots A_k B_1 \cdots B_s A_s A_{s+1} \\
& \quad A_{s+2} \cdots A_k \\
\Leftrightarrow A_s \cdots A_k B_1 \cdots B_k &= A_{s+1} A_s A_{s+2} A_{s+1} A_{s+3} A_{s+2} A_{s+4} A_{s+3} A_{s+5} \\
& \quad A_{s+4} \cdots A_{k-1} A_{k-2} A_k A_{k-1} \\
& \quad B_1 A_k B_2 \cdots B_s \\
\Leftrightarrow A_{k-m} \cdots A_k B_1 \cdots B_k &= A_{k-m+1} A_s A_{k-m+2} A_{s+1} A_{k-m+3} A_{k-m+2} \\
& \quad A_{k-m+4} A_{k-m+3} A_{k-m+5} A_{k-m+4} \cdots A_{k-1} \\
& \quad A_{k-2} A_k A_{k-1} B_1 A_k B_2 \cdots B_{k-m}
\end{aligned}$$

Therefore, claims 6.2.2 and 6.2.3 hold for all $m < k$, by induction. It remains to show that claim 6.2.3 is true when $m = k$. But this is obvious since, in this case, $(xy)^m x = (yx)^m y \Leftrightarrow A_1 \cdots A_k = A_1 \cdots A_k$.

Now we prove theorem 6.2.1, which is an easy consequence of the two claims. For the sufficient condition, suppose $n = t(2k+1)$, $t \in \mathbb{N}$. It suffices to show $l = 2k+1 \Rightarrow prod(x, y; l) = prod(y, x; l)$. Clearly, $m = k$ when $l = 2k+1$. Therefore, it follows by, claim 6.2.3, that

$$prod(x, y; l) = prod(y, x; l) \Leftrightarrow A_1 \cdots A_k = A_{k-k+1} \cdots A_k B_1 \cdots B_{k-k}$$

$$\Leftrightarrow A_1 \cdots A_k = A_1 \cdots A_k$$

For the necessary condition, assume that $2k+1 \nmid l$. Then, $prod(x,y;l) = prod(y,x;l)$ if and only if $prod(x,y;r) = prod(y,x;r)$ for some $r \in \{1,2,\cdots,2k\}$, where $l \equiv r(mod(2k+1))$.

If r is even, then by claim 6.2.2, $prod(x,y;r) = prod(y,x;r) \Leftrightarrow$

$$\begin{aligned} A_{k-q+1}\cdots A_k B_1 \cdots B_k &= A_{k-q+2}A_{k-q+1}A_{k-q+3}A_{k-q+2}A_{k-q+4}A_{k-q+3}A_{k-q+5} \\ & \quad A_{k-q+4}\cdots A_{k-1}B_1 A_k B_2 \cdots B_{k-q+1} \end{aligned}$$

where $q = \frac{r}{2}$. Note that since $r \leq 2k$, this equation holds for $q \leq k$, $k \geq 2$.

$$\begin{aligned} LHS(a_{k-q+1}) &= A_{k-q+1}A_{k-q+2}(a_{k-q+1}) \\ &= a_{k-q+2} \end{aligned}$$

$$\begin{aligned} RHS(a_{k-q+1}) &= A_{k-q+2}A_{k-q+1}A_{k-q+3}A_{k-q+2}(a_{k-q+1}) \\ &= A_{k-q+2}A_{k-q+3}A_{k-q+1}A_{k-q+2}(a_{k-q+1}) \\ &= A_{k-q+2}A_{k-q+3}(a_{k-q+2}) \\ &= a_{k-q+3} \\ &\neq a_{k-q+2} \end{aligned}$$

Hence, LHS $\neq$ RHS.

If r is odd, then $prod(x,y;r) = prod(y,x;r) \Leftrightarrow A_1\cdots A_k = A_{k-p+1}\cdots A_k B_1 \cdots B_{k-p}$, where $p = \frac{r-1}{2}$. But then, provided that $k-p+1 > 1$, we have $LHS(a_1) = A_1A_2(a_1) = a_2$, while $RHS(a_1) = a_1 \neq a_2$. Hence, LHS $\neq$ RHS.

It remains to show that $k-p+1 > 1$. Indeed, $p \geq k \Rightarrow \frac{r-1}{2} \geq k \Rightarrow r \geq 2k+1$, contradicting $r \in \{1,\cdots,2k\}$. $\square$

6.3 Artin relations from foldings

In this section, we use the theory of Artin groups to find Artin relations in the mapping class group. Even more, we give explicit elements x and y in $Mod(S)$ that generate Artin groups of types $I_2(k)$ ($k \geq 3$) and $I_2(2k-2)$ ($k \geq 4$). To do that, we invoke LCM-homomorphisms (described in section 3.2) induced by the dihedral foldings $A_{k-1} \to I_2(k)$ and $D_k \to I_2(2k-2)$. The induced embeddings between the corresponding Artin groups provide k and $2k-2$ Artin relations in $\mathcal{A}(A_{k-1})$ and $\mathcal{A}(D_k)$ respectively. Since the Artin groups of types A_{k-1} and D_k inject into the corresponding mapping class groups via the geometric homomorphism, we obtain Artin relations of length k and $2k-2$ in $Mod(S_\Gamma)$, $\Gamma = A_{k-1}, D_k$.

Theorem 6.3.1. *Let $k \geq 3$ be an integer. Suppose that $a_1, a_2, \cdots, a_{k-1}$ form a $(k-1)$-chain in $S_{A_{k-1}}$. Let*

$$x = \begin{cases} T_1 T_3 \cdots T_{k-3} T_{k-1} & \text{when } k \text{ is even} \\ T_1 T_3 \cdots T_{k-4} T_{k-2} & \text{when } k \text{ is odd} \end{cases}$$

$$y = \begin{cases} T_2 T_4 \cdots T_{k-4} T_{k-2} & \text{when } k \text{ is even} \\ T_2 T_4 \cdots T_{k-3} T_{k-1} & \text{when } k \text{ is odd} \end{cases}$$

Then x and y generate the Artin group $\mathcal{A}(I_2(k))$ in $Mod(S_{A_{k-1}})$. Moreover, $prod(x, y; n) = prod(y, x; n)$ if and only if $n \equiv 0 mod(k)$.

Proof. We only prove the case when k is even. The odd case is proved similarly. Assume k is an even integer greater than 3. Consider the Coxeter graphs A_{k-1} and $I_2(k)$, and label their vertices by the sets $P = \{s_1, s_2, \cdots, s_{k-1}\}$ and $Q = \{s, t\}$ respectively. Partition P into $K_s = \{s_1, s_3, \cdots, s_{k-3}, s_{k-1}\}$ and $K_t = \{s_2, s_4, \cdots, s_{k-4}, s_{k-2}\}$. By corollary 3.3.4, the dihedral folding $f : A_{k-1} \to I_2(k)$ such that $f(K_s) = s$ and $f(K_t) = t$ induces the LCM-homomorphism

$$\phi^f : \mathcal{A}^+(I_2(k)) \to \mathcal{A}^+(A_{k-1})$$

$$s \mapsto \Delta_{f^{-1}(s)}$$

$$t \mapsto \Delta_{f^{-1}(t)}$$

By Lemma 5.2.1, $\Delta_{f^{-1}(s)} = s_1 s_3 \cdots s_{k-3} s_{k-1}$ and $\Delta_{f^{-1}(t)} = s_2 s_4 \cdots s_{k-4} s_{k-2}$. Corollary 3.3.4 implies that ϕ^f is injective. By theorem 3.2.3, ϕ^f induces an injective homomorphism ϕ between the corresponding Artin groups. The curve graph associated to the a_i is isomorphic to A_{k-1}. Consequently, the geometric homomorphism

$$g : \mathcal{A}(A_{k-1}) \to Mod(S_{A_{k-1}})$$

$$s_i \mapsto T_i$$

is injective by theorem 5.1.2. As such, the composition $g \circ \phi$ gives a monomorphism of $\mathcal{A}(I_2(k))$ into $Mod(S_{A_{k-1}})$.

Since x and y generate $\mathcal{A}(I_2(k))$, $prod(x, y; k) = prod(y, x; k)$. From this equality, it follows immediately that $prod(x, y; pk) = prod(y, x; pk)$ for all positive integers p. This proves the sufficient condition of the last statement in theorem 6.3.1. For the necessary condition, assume $k \nmid n$ and $prod(x, y; n) = prod(y, x; n)$. Then $prod(x, y; r) = prod(y, x; r)$ for some $r \in \{1, \cdots, k-1\}$. But since $g \circ \phi$ is injective, this would mean that $prod(s, t; r) = prod(t, s; r)$ in $\mathcal{A}(I_2(k))$, which is a contradiction. □

Theorem 6.3.2. *Let $k \geq 4$ be an integer, and suppose that curves $a_1, a_2, \cdots, a_k$ have curve graph D_k in S_{D_k}. Let*

$$x = \begin{cases} T_1 T_3 \cdots T_{k-3} T_{k-1} T_k & \text{when } k \text{ is even} \\ T_1 T_3 \cdots T_{k-2} & \text{when } k \text{ is odd} \end{cases}$$

$$y = \begin{cases} T_2 T_4 \cdots T_{k-2} & \text{when } k \text{ is even} \\ T_2 T_4 \cdots T_{k-3} T_{k-1} T_k & \text{when } k \text{ is odd} \end{cases}$$

Then x and y generate the Artin group $\mathcal{A}(I_2(2k-2))$ in $Mod(S_{D_k})$. Moreover, $prod(x, y; n) = prod(y, x; n)$ if and only if $n \equiv 0 mod (2k-2)$.

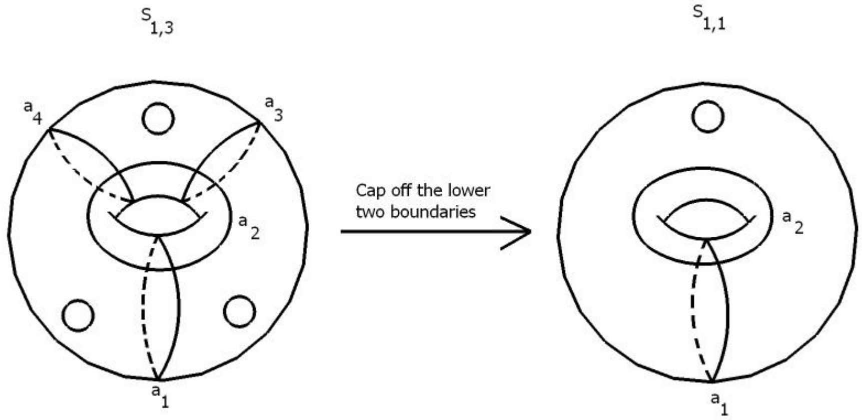


Figure 6.3: If $x = T_1T_3T_4$ and $y = T_2$, it follows from theorem 6.3.2 that $(xy)^3 = (yx)^3$ in $Mod(S_{1,3})$. By capping off the two boundary components, one gets the relation $(T_1^3T_2)^3 = (T_2T_1^3)^3$ in $Mod(S_{1,1})$.

Proof. Again, we only prove the even case. The odd one is proved the same way. Label the vertices of D_k and $I_2(2k-2)$ by $P = \{s_1, s_2, \cdots, s_k\}$ and $Q = \{s, t\}$ respectively. Partition P into $K_s = \{s_1, s_3, \cdots, s_{k-1}, s_k\}$ and $K_t = \{s_2, s_4, \cdots, s_{k-2}\}$. Both K_s and K_t consist of pairwise commuting generators of $\mathcal{A}^+(D_k)$. By Lemma 5.2.1, the least common multiple of each of these sets is the product of its elements (in any order). The dihedral folding $f : D_k \to I_2(2k-2)$ induces the LCM-homomorphism $\phi^f : \mathcal{A}^+(I_2(2k-2)) \to \mathcal{A}^+(D_k)$, which maps

$$s \mapsto \Delta_{f^{-1}(s)} = s_1s_3 \cdots s_{k-1}s_k$$

$$t \mapsto \Delta_{f^{-1}(t)} = s_2s_4 \cdots s_{k-2}$$

Since ϕ^f is injective, the induced map, ϕ, on the corresponding Artin group is injective as well. By post-composing with the geometric homomorphism $g : \mathcal{A}(D_k) \to Mod(S_{D_k})$, one gets an embedding of $I_2(2k-2)$ into $Mod(S_{D_k})$. This produces a subgroup of $Mod(S_{D_k})$ which is isomorphic to the Artin group $A(I_2(2k-2))$, and is generated by x and y. Since $prod(s, t; 2k-2) = prod(t, s; 2k-2)$, it follows that $prod(x, y; 2k-2) = prod(y, x; 2k-2)$. The rest of the proof follows as in theorem 6.3.1. $\square$

Corollary 6.3.3 (Corollary to theorem 6.3.2)**.** *Let a_1 and a_2 be isotopy classes of simple closed curves in S. If $i(a_1, a_2) = 1$, then T_1^3 and T_2 satisfy an Artin relation of length 6 in $Mod(S)$.*

Proof. Let F be a regular neighborhood of $a_1 \cup a_2$ so that F is homeomorphic to $S_{1,1}$. In F, consider three parallel copies of a_1 denoted by a_1, a_3, and a_4. Now remove two open disks from F to obtain $S_{1,3}$ and curves a_1, a_2, a_3, and a_4 as in Figure 6.3.

When $k = 4$, theorem 6.3.2 implies that $x = T_1T_3T_4$ and $y = T_2$ satisfy $xyxyxy = yxyxyx$ in $Mod(S_{D_4}) = Mod(S_{1,3})$. By theorem 5.1.2, the subgroup G of $Mod(S_{1,3})$ generated by T_1, T_2, T_3, and T_4 is isomorphic to $\mathcal{A}(D_4)$. Now, reverse the process and cap off the two boundary components to recover $F \approx S_{1,1}$. There is a homomorphism $G \to Mod(S_{1,1})$ defined by $T_2 \mapsto T_2$ and $T_j \mapsto T_1$ for $j = 1, 3, 4$. The image of $xyxyxy = yxyxyx$ under this homomorphism is $(T_1^3T_2)^3 = (T_2T_1^3)^3$ in $Mod(S_{1,1})$. Of course, the same relation is true in $Mod(S)$. $\square$

CHAPTER 7

SUBGROUPS OF MOD(S) GENERATED BY THREE DEHN TWISTS

7.1 Introduction

In this chapter, we are going to study subgroups of $Mod(S)$ generated by three Dehn twists. Suppose that a_1, a_2, and a_3 are essential, pairwise nonisotopic simple closed curves in S. As alluded in chapter 1, we shall not distinguish between a simple closed curve and its isotopy class notationally.

Consider the isotopy classes a_1, a_2, and a_3, and assume that all the geometric intersections $i(a_j, a_k) \in \{0, 1, 2\}$. This assumption will keep the combinatorics manageable to a certain extent. Denote by T_1, T_2, and T_3 the respective (left) Dehn twists along a_1, a_2 and a_3. If G denotes the subgroup of $Mod(S)$ generated by T_1, T_2, and T_3, we find presentations for G corresponding to different configurations of the simple closed curves a_1, a_2, and a_3. We remark that the question of determining subgroups of $Mod(S)$ generated by three Dehn twists is considerably harder than the case of two Dehn twists. It is a nontrivial task to show that a non-obvious defining relation exists between the T_i in G. It is also very hard to prove that no such defining relations exist.

The structure of G could depend on the surface in which the simple closed curves a_1, a_2 and a_3 are viewed. For instance, let S be a compact orientable surface containing a_1, a_2, and

a_3, and denote by N_ϵ a closed regular neighborhood of $a_1 \cup a_2 \cup a_3$. If S is obtained from N_ϵ by capping off some boundary component of N_ϵ with a disk, this could possibly add more defining relations between the T_i, $i = 1, 2, 3$, in $Mod(S)$. As such, G viewed as a subgroup of $Mod(N_\epsilon)$ and G viewed as a subgroup of $Mod(S)$ are not necessarily isomorphic. Hence, it is important to specify the ambient group $Mod(S)$ when studying G.

In almost all the subsequent sections, we shall carry out our analysis as follows. First, we study the structure of G viewed as a subgroup of $Mod(N_\epsilon)$, where N_ϵ is a closed regular neighborhood of $a_1 \cup a_2 \cup a_3$. Then, we use theorem 1.5.2 to determine G as a subgroup of $Mod(S)$, where S contains N_ϵ as a subsurface. Since N_ϵ is the smallest compact surface containing a_1, a_2, and a_3, this covers all possible compact (with possibly empty boundary) surfaces containing the a_i, $i = 1, 2, 3$. As a result, we obtain all the possible structures of G corresponding to a given configuration of a_1, a_2, and a_3.

Our program for understanding subgroups of $Mod(S)$ generated by three Dehn twists depends on the geometric intersections $i(a_j, a_k)$, $j < k$ and $j, k \in \{1, 2, 3\}$. Clearly, there are three such geometric intersections. Following Dickinson's notation in [8], we encapsulate these three geometric intersections in an ordered triple as follows. Given $i(a_1, a_2) = x_{12}$, $i(a_1, a_3) = x_{13}$, and $i(a_2, a_3) = x_{23}$, we use the ordered triple (x_{12}, x_{13}, x_{23}). This ordered triple shall henceforth encode the geometric intersections $i(a_j, a_k)$ with the above defined order.

In section 7.8, we shall amend the triple notation slightly to account for isotopy classes a_1, a_2, and a_3 whose intersection triple is $(2, 1, 0)$, but whose corresponding closed regular neighborhoods $N_\epsilon = N_\epsilon(a_1 \cup a_2 \cup a_3)$ are non-homeomorphic. The reason we do this is because the structure of G might not be the same in each case. We shall see in subsections 7.8.1 and 7.8.2 that there are two non-homeomorphic surfaces N_ϵ associated with the intersection triple $(2, 1, 0)$. These surfaces can be distinguished by examining the algebraic intersection number $\hat{i}(a_1, a_2)$ (see section 1.1).

Recall that when $i(a_j, a_k) = 1$, $\hat{i}(a_j, a_k) = \pm 1$. In this case, a closed regular neighborhood of $a_j \cup a_k$ is homeomorphic to $S_{1,1}$. On the other hand, when $i(a_j, a_k) = 2$, either $\hat{i}(a_j, a_k) = \pm 2$ or 0. When $\hat{i}(a_j, a_k) = 0$, a closed regular neighborhood of $a_j \cup a_k$ is homeomorphic to $S_{0,4}$, whereas $\hat{i}(a_j, a_k) = \pm 2$ implies that a closed regular neighborhood of $a_j \cup a_k$ is homeomorphic to $S_{1,2}$. In particular, depending on whether $\hat{i}(a_1, a_2) = 0$ or ± 2, the intersection triple $(2, 1, 0)$ gives rise to two closed regular neighborhoods $N_\epsilon(a_1 \cup a_2 \cup a_3)$ given by $S_{2,1}$ and $S_{1,3}$ respectively (see subsections 7.8.1 and 7.8.2). Thus, we amend the notation $(2, 1, 0)$ and write $(2_\gamma, 1, 0)$ where $\gamma = \hat{i}(a_1, a_2)$.

It is easy to see that there is a one to one correspondence between the set $\{(x_{12}, x_{13}, x_{23})\}$ of intersection triples and the set of (non-isomorphic) curve graphs of a_1, a_2, and a_3. Since we are assuming $i(a_j, a_k) \in \{0, 1, 2\}$, there are ten distinct intersection triples corresponding to $(0, 0, 0)$, $(1, 0, 0)$, $(2, 0, 0)$, $(1, 0, 1)$, $(1, 1, 1)$, $(2, 1, 0)$, $(2, 2, 0)$, $(2, 1, 1)$, $(2, 2, 1)$, and $(2, 2, 2)$. As such, there are ten non-isomorphic curve graphs corresponding to these triples. These graphs are shown in Table 7.1.

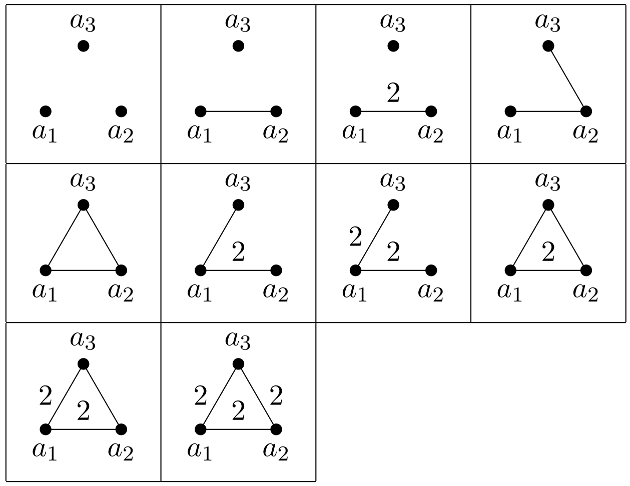
In the subsequent sections, we shall find explicit presentations for G (viewed as a subgroup of $Mod(S)$ for all possible S) for the first five intersection triples. For the triple $(2, 1, 0)$, we study G corresponding to $(2_{\pm 2}, 1, 0)$ and $(2_0, 1, 0)$. In these cases, we are not able to find an explicit presentation for G. However, we show (with the help of the computer algebra software Magma) that G is finitely presented and is isomorphic to an infinite index subgroup of some Artin group. For the remaining four triples, we have obtained some partial results similar to those of section 7.8, but we do not include them in the dissertation.

7.2 Subgroups of Mod(S) generated by two Dehn twists

Before delving into our study of subgroups generated by three Dehn twists, we recall the subgroups of $Mod(S)$ generated by two Dehn twists. Consider two simple closed curves a and b in S. The subgroups of $Mod(S)$ generated by the Dehn twists T_a and T_b are classified according to the geometric intersection number $i(a, b)$. Precisely,

Theorem 7.2.1. *If $i(a, b) = 0$, then $\langle T_a, T_b\rangle \cong \mathbb{Z} \oplus \mathbb{Z}$.*

Table 7.1: This table shows the ten (non-isomorphic) curve graphs determined by $\{(x_{12}, x_{13}, x_{23}) | x_{jk} \in \{0, 1, 2\}\}$.



Theorem 7.2.2. *If $i(a, b) = 1$, then $\langle T_a, T_b\rangle \cong SL_2(\mathbb{Z})$ when $S = S_{1,0}$, and $\langle T_a, T_b\rangle \cong \mathcal{B}_3$ otherwise.*

Theorem 7.2.3. *If $i(a, b) \geq 2$, then $\langle T_a, T_b\rangle \cong \mathbb{F}_2$.*

Theorem 7.2.1 follows immediately from proposition 1.3.7 and fact 1.3.6. It is also a special case of theorem 7.3.3. When $S \neq S_{1,0}$ and $i(a, b) = 1$, $\langle T_a, T_b\rangle \cong \mathcal{B}_3$ follows easily from theorem 5.1.2 and corollary 1.5.3. When $S = S_{1,0}$, $\langle T_a, T_b\rangle$ has the additional relation $(T_aT_b)^6 = 1$, which makes it isomorphic to $SL_2(\mathbb{Z})$. Finally, note that theorem 7.2.3 is theorem 1.3.11, due to Ishida [15].

7.3 The case $(0, 0, 0)$

A more general result of this case is well known. More precisely, if $\{a_1, \cdots, a_n\}$ is a collection of essential, pairwise nonisotopic, and pairwise disjoint simple closed curves in S, then the subgroup of $Mod(S)$ generated by the $T_j, i = 1, \cdots, n$ is isomorphic to the free abelian group of rank n. A nice proof, which we include here for completeness, can be found in [23]. The proof makes use of the following two lemmas.

Lemma 7.3.1 (Rolfsen-Paris). *Suppose that F is an essential subsurface of S, and let a, b be essential simple closed curves in F. Assume that a is not isotopic in F to a boundary component of an exterior cylinder (see definition 1.5.1). Then a and b are isotopic in F if and only if they are isotopic in S.*

Lemma 7.3.2 (Rolfsen-Paris). *Suppose $a_1, \cdots, a_p$ are essential simple closed curves in S such that:*

- *$a_j \cap a_k = \varnothing$, if $j \neq k$,*
- *a_j is not isotopic to a_k, if $j \neq k$,*
- *none of the a_j is peripheral (ie isotopic to a boundary component) in S.*

Then for each $j, 1 \leq j \leq p$, there exists an isotopy class of simple closed curves b such that $i(a_k, b) = 0$ if $j \neq k$, and $i(a_j, b) > 0$.

Theorem 7.3.3. *Suppose that $\{a_1, \cdots, a_n\}$ is a collection of essential, pairwise disjoint simple closed curves in S. Assume that, for all $j \neq k$, a_j is not isotopic to a_k. Then the subgroup G of $Mod(S)$ generated by the Dehn twists $T_1, \cdots, T_n$ is isomorphic to $\mathbb{Z}^n$.*

Proof. Let $w \in G$. Since all the T_j commute with one another, we may write $w = T_1^{m_1} T_2^{m_2} \cdots T_n^{m_n}$, $m_i \in \mathbb{Z}$. Define the map $\tau : \mathbb{Z}^n \to Mod(S)$ by

$$\tau(m_1, m_2, \cdots, m_n) = T_1^{m_1} T_2^{m_2} \cdots T_n^{m_n}$$

Since the T_j are pairwise commutative, τ is a homomorphism. To prove injectivity, suppose $w = T_1^{m_1} T_2^{m_2} \cdots T_n^{m_n}$ is equal to the identity in $Mod(S)$. Let $\hat{S}$ be the closed surface obtained from S by gluing $S_{1,1}$ to each boundary component of S. The a_j are still essential in $\hat{S}$, and lemma 7.3.1 implies that a_j is not isotopic to a_k in $\hat{S}$, for $j \neq k$. Fix an arbitrary $j \in \{1, 2, \cdots, n\}$. By lemma 7.3.2, there is a simple closed curve b such that $i(a_j, b) > 0$ and $i(a_k, b) = 0$ for all $k \neq j$. This implies that $[T_k, T_b] = 1$ for all $k \neq j$. Consequently, $b = w(b) = T_1^{m_1} T_2^{m_2} \cdots T_n^{m_n}(b) = T_j^{m_j}(b)$. By fact 1.3.5, we have

$$0 = i(b, b) = i(T_j^{m_j}(b), b) = |m_j|(a_j, b)^2$$

Since $i(a_j, b) > 0$, $m_j = 0$. And since j was arbitrary, $m_j = 0$ for all j. Now we have

$$\mathbb{Z}^n \xrightarrow{\tau} Mod(S) \xrightarrow{i_*} Mod(\hat{S})$$

where $i_* \circ \tau$ is injective. Therefore, τ is injective. $\square$

7.4 The case $(1, 0, 0)$

let a_1, a_2, and a_3 be distinct isotopy classes of essential simple closed curves in $S = S_{g,b}$, satisfying the triple $(1, 0, 0)$. For this to happen, it must be the case that $(g, b) \in \mathbb{N} \times \mathbb{N}_{\geq 0} \backslash \{(1, 0)\}$. Since $i(a_1, a_2) = 1$ and $S_{1,0}$ is excluded, theorem 7.2.2 implies that $\langle T_1, T_2 \rangle \cong \mathcal{B}_3$ in $Mod(S)$. The following lemma, due to Margalit [22], characterizes the 2-chain relation in $Mod(S)$.

Lemma 7.4.1 (Margalit). *Suppose $M = (T_x T_y)^k$, where M is a multitwist word (ie product of Dehn twists along pairwise disjoint curves) and $k \in \mathbb{Z}$, is a nontrivial relation in $Mod(S)$, and $[M, T_x] = 1$. Then the given relation is the 2-chain relation. This means that $M = T_c^j$, where $c = \partial N_\epsilon(x \cup y)$, and $k = 6j$.*

Theorem 7.4.2. *Suppose that a_1, a_2, and a_3 are distinct isotopy classes of essential simple closed curves in S, satisfying the triple $(1, 0, 0)$. Let G be the subgroup of $Mod(S)$ generated by T_1, T_2, and T_3. Then*

- *$G \cong \mathcal{B}_3 \times \mathbb{Z}$ if $a_3 \neq \partial N_\epsilon(a_1 \cup a_2)$*
- *$G \cong \mathcal{B}_3$ if $a_3 = \partial N_\epsilon(a_1 \cup a_2)$*

Proof. Assume that $a_3 \neq \partial N_\epsilon(a_1 \cup a_2)$. Define the map

$$\tau : \langle T_1, T_2 \rangle \times \mathbb{Z} \to G = \langle T_1, T_2, T_3 \rangle$$
$$(f, n) \mapsto f T_3^n$$

We show that τ is an isomorphism. Clearly, τ is well-defined. It is also obvious that τ is surjective.

$$\begin{aligned} \tau\left((f_1, n_1)(f_2, n_2)\right) &= \tau\left(f_1.f_2, n_1 + n_2\right) \\ &= f_1.f_2 T_3^{n_1+n_2} \\ &= (f_1 T_3^{n_1})(f_2 T_3^{n_2}) \\ &= \tau\left(f_1, n_1\right) \tau\left(f_2, n_2\right) \end{aligned}$$

The third equality is due to the fact that T_3 commutes with both T_1 and T_2. It remains to show the injectivity of τ. If $fT_3^k = gT_3^n$, where $f, g \in \langle T_1, T_2 \rangle$ and $k, n \in \mathbb{Z}$, then $g^{-1}f = T_3^{n-k}$. $g^{-1}f$ is an element of $\langle T_1, T_2 \rangle \cong \mathcal{B}_3$. Set $w = g^{-1}f$. Since $[w, T_i] = 1$ for $i = 1, 2$, w is in the center of $\langle T_1, T_2 \rangle$. By theorem 2.2.1, the center of $\langle T_1, T_2 \rangle$ is generated by $(T_1T_2)^3$. As such, $w = (T_1T_2)^{3p}$ for some $p \in \mathbb{Z}$. Now, $(T_1T_2)^{3p} = T_3^{n-k}$ is a Dehn twist relation where T_3^{n-k} is a multitwist, and $[T_3^{n-k}, T_1] = 1$. By lemma 7.4.1, this relation is either the 2-chain relation or a trivial relation. The first case implies $a_3 = \partial N_\epsilon(a_1 \cup a_2)$, which contradicts the hypothesis. Hence, $g^{-1}f = T_3^{n-k}$ must be trivial $Mod(S)$. This gives $n = k$ and $f = g$.

Now suppose that $a_3 = \partial N_\epsilon(a_1 \cup a_2)$. By the chain relation, $T_3 = (T_1T_2)^6$. Hence, $G = \langle T_1, T_2, T_3 \rangle$ reduces to $\langle T_1, T_2 \rangle \cong \mathcal{B}_3$, the classical braid group on three strands. □

7.5 The case $(2, 0, 0)$

Theorem 7.5.1. *Suppose a_1, a_2, and a_3 are distinct isotopy classes of essential simple closed curves satisfying the triple $(2, 0, 0)$ in S. If G is the subgroup of $Mod(S)$ generated by the Dehn twists T_1, T_2, and T_3, the $G \cong \mathbb{F}_2 \times \mathbb{Z}$.*

Proof. Define the map

$$\tau : \langle T_1, T_2 \rangle \times \mathbb{Z} \to G = \langle T_1, T_2, T_3 \rangle$$
$$(f, n) \mapsto fT_3^n$$

τ is clearly well-defined. That τ is surjective follows from the fact that every element of G may be written in the form fT_3^n, where $f \in \langle T_1, T_2 \rangle$ and $n \in \mathbb{Z}$. This is because T_3 commutes with both T_1 and T_2.

$$\begin{aligned} \tau((f_1, n_1)(f_2, n_2)) &= \tau(f_1.f_2, n_1 + n_2) \\ &= f_1.f_2 T_3^{n_1+n_2} \\ &= (f_1 T_3^{n_1})(f_2 T_3^{n_2}) \\ &= \tau(f_1, n_1)\, \tau(f_2, n_2) \end{aligned}$$

For injectivity, suppose that $fT_3^p = gT_3^q$ for some $f, g \in \langle T_1, T_2 \rangle$ and $p, q \in \mathbb{Z}$. This equality holds if and only if $g^{-1}f = T_3^{q-p}$. Set $w = g^{-1}f$ and $l = q - p$. We will show that $w = T_3^l$ implies $l = 0$. Consequently, $p = q$ and $f = g$.

Since $i(a_1, a_2) \geq 2$, T_1 and T_2 generate a free group according to theorem 7.2.3. Assume $w = T_3^l$. Since $i(a_1, a_3) = 0$, $[w, T_1] = 1$. But, $w \in \langle T_1, T_2 \rangle \cong \mathbb{F}_2$. So, $w = T_1^r$ for some $r \in \mathbb{Z}$. This implies that $T_1^r = T_3^l$. Since $a_1 \neq a_3$, corollary 1.3.10 implies that $l = r = 0$. □

We remark that the proof of theorem 7.5.1 establishes the following stronger result.

Theorem 7.5.2. *For every integer $m \geq 2$, if a_1, a_2 and a_3 satisfy the triple $(m, 0, 0)$ in S, then the subgroup of $Mod(S)$ generated by the Dehn twists T_1, T_2, and T_3 is isomorphic to $\mathbb{F}_2 \times \mathbb{Z}$.*

7.6 The case $(1, 0, 1)$

Suppose that a_1, a_2 and a_3 are distinct isotopy classes of simple closed curves in S, satisfying the intersection triple $(1, 0, 1)$. As illustrated in Table 7.1, the curve graph induced by the a_i is the Coxeter graph A_3. By part (ii) of theorem 4.2.1, the surface S_{A_3} is homeomorphic to $S_{1,2}$. Recall that S_{A_3} represents a closed regular neighborhood of $a_1 \cup a_2 \cup a_3$. Let G denote the subgroup of $Mod(S_{1,2})$ generated by T_1, T_2, and T_3. By theorem 5.1.2, G is isomorphic to $\mathcal{B}_4$. $S_{1,2}$ is a subsurface of $S_{g,b}$ for all $(g, b) \in \mathcal{X}$, where

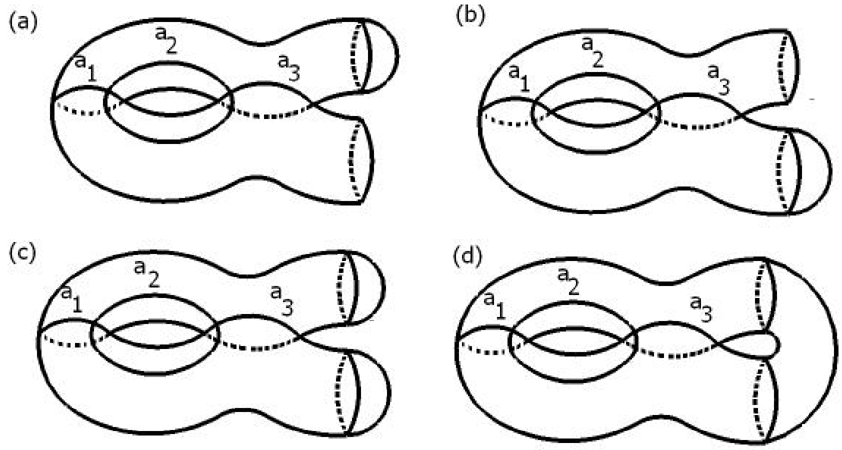


Figure 7.1: Capping off $\partial S_{1,2}$ with disks or with an exterior cylinder.

$$\mathcal{X} = \{(p,q) \in \mathbb{Z} \times \mathbb{Z} : p \geq 1, q \geq 0\}$$

By corollary 1.5.3, the homomorphism $i_* : Mod(S_{1,2}) \to Mod(S_{g,b})$ is injective for all $(g,b) \in \mathcal{X} \backslash \{(1,0),(1,1),(2,0)\}$. In other words, i_* is injective except when $S = S_{1,1}$, $S_{1,0}$, or $S_{2,0}$. The surface $S_{1,1}$ is obtained from $S_{A_3} \approx S_{1,2}$ when either boundary component is capped off with a disk (figure 7.1 (a)&(b)). $S_{1,0}$ is obtained from S_{A_3} by capping off both boundary components with disks, as in figure 7.1 (c). Finally, $S_{2,0}$ is obtained from S_{A_3} by attaching an exterior cylinder as in figure 7.1 (d).

In the cases (a), (b), and (c) corresponding to $S_{1,1}$ and $S_{1,0}$, $a_1 = a_3$, violating the assumption that they must be distinct isotopy classes. So, in these cases, the triple $(1,0,1)$ is not satisfied to start with. In case (d), we know that $i_* : Mod(S_{1,2}) \to Mod(S_{2,0})$ is not injective. We shall prove, however, that $i_*|_G$ is injective.

By theorem 1.5.2, $ker(i_*)$ is normally generated by $T_4T_5^{-1}$, where a_4 and a_5 are the peripheral (ie boundary parallel) curves in $S_{1,2}$. Since T_4 and T_5 are both central in $Mod(S_{1,2})$, so is $T_4T_5^{-1}$. Consequently, $ker(i_*)$ is generated by $T_4T_5^{-1}$.

G injects into $Mod(S_{2,0})$ if $G \cap ker(i_*) = \{1\}$. In other words, there does not exist a nontrivial element $W \in G$ and $n \in \mathbb{Z}\backslash\{0\}$ such that $W = (T_4T_5^{-1})^n$. We will now prove that this is indeed the case. Suppose that $W = (T_4T_5^{-1})^n$ in $Mod(S_{1,2})$ for some $n \in \mathbb{Z}\backslash\{0\}$.

As T_4 and T_5 commute with all the $T_i, i = 1, 2, 3$, W is in the center, $Z(G)$, of G. Since G is isomorphic to $\mathcal{B}_4$, it follows that $Z(G)$ is infinite cyclic, generated by $(T_1T_2T_3)^4$ according to theorem 2.2.1. Hence, $W = (T_1T_2T_3)^{4k}$ for some $k \in \mathbb{Z}$. On the other hand, the chain relation gives $T_4T_5 = (T_1T_2T_3)^4$, and so $(T_4T_5)^k = (T_1T_2T_3)^{4k}$. Noting that $[T_4, T_5] = 1$, we have

$$\begin{aligned} T_4^n T_5^{-n} &= T_4^k T_5^k \Leftrightarrow \\ T_4^{n-k} &= T_5^{n+k} \end{aligned}$$

Since $a_4 \neq a_5$ in $S_{1,2}$, it follows from corollary 1.3.10 that $n = k = 0$, a contradiction. We have proved the following theorem:

Theorem 7.6.1. *Let a_1, a_2, and a_3 be distinct isotopy classes of simple closed curves satisfying the triple $(1, 0, 1)$ in S. If $G < Mod(S)$ is generated by the Dehn twists T_1, T_2, and T_3, then G is isomorphic to $\mathcal{B}_4$. More explicitly, G has presentation*

$$\langle T_1, T_2, T_3 \mid T_1T_2T_1 = T_2T_1T_2,\ T_1T_3 = T_3T_1,\ T_2T_3T_2 = T_3T_2T_3 \rangle$$

7.7 The case $(1, 1, 1)$

Consider three distinct isotopy classes a_1, a_2, and a_3 of simple closed curves in S satisfying the triple $(1, 1, 1)$. Let T_i, $i = 1, 2, 3$ represent the (left) Dehn twist along a_i, and denote by G the subgroup of $Mod(S)$ generated by T_1, T_2, T_3. In this section, we study the structure of G, viewed as a subgroup of $Mod(S)$ for all possible S. We shall make use of the following theorem by Charney and Peifer [5].

Theorem 7.7.1 (Charney-Peifer)**.** *The center $Z\mathcal{A}(\widetilde{A}_{n-1})$ of $\mathcal{A}(\widetilde{A}_{n-1})$ is trivial.*

Theorem 7.7.2. *Suppose that a_1, a_2, and a_3 are distinct isotopy classes of simple closed curves in S, satisfying the triple $(1, 1, 1)$. Let $\widetilde{S}$ represent a closed regular neighborhood of $a_1 \cup a_2 \cup a_3$. By theorem 4.3.4, $\widetilde{S}$ is homeomorphic to $S_{1,3}$ and one of the boundary* *components is distinguished in the sense of definition 4.3.5. Denote by T_i the (left) Dehn twist along a_i and by G the subgroup of $Mod(S)$ generated by T_1, T_2, and T_3. Then the structure of G is given as follows:*

(1) *If $S = \widetilde{S}$ (ie $S = S_{1,3}$), then $G \cong \mathcal{A}(\widetilde{A}_2)$.*

(2) *If S contains $\widetilde{S}$ as a subsurface and no component of no component of $\overline{S \backslash \widetilde{S}}$ is a cylinder exterior to $\widetilde{S}$ or a disk with less than two punctures, then $G \cong \mathcal{A}(\widetilde{A}_2)$.*

(3) *If S is obtained from $\widetilde{S}$ by capping off one boundary component c_i with a disk (ie $S = S_{1,2}$), then $G \cong \mathcal{A}(\widetilde{A}_2)$ when c_i is distinguished, and $G \cong \mathcal{B}_4$ otherwise.*

(4) *If S is obtained from $\widetilde{S}$ by capping off two boundary components with disks (ie $S = S_{1,1}$), then $G \cong \mathcal{B}_3$.*

(5) *If S is obtained from $\widetilde{S}$ by capping off three boundary components with disks (ie $S = S_{1,0}$), then $G \cong SL_2(\mathbb{Z})$.*

(6) *If S is obtained from $\widetilde{S}$ by attaching an exterior cylinder (ie $S = S_{2,1}$), then $G \cong \mathcal{A}(\widetilde{A}_2)$.*

(7) *If S is obtained from $\widetilde{S}$ by attaching an exterior cylinder and capping off the remaining boundary component c_i with a disk (ie $S = S_{2,0}$), then $G \cong \mathcal{A}(\widetilde{A}_2)$ when c_i is distinguished, and $G \cong \mathcal{B}_4$ otherwise.*

Proof. Assume that the isotopy classes a_1, a_2, and a_3 satisfy the triple $(1,1,1)$ in S, and consider the subsurface $\widetilde{S}$, which is a closed regular neighborhood of $a_1 \cup a_2 \cup a_3$. By theorem 5.5.4, $\mathcal{A}(\widetilde{A}_2)$ is isomorphic to G via the geometric homomorphism sending the i^{th} standard generator γ_i of $\mathcal{A}(\widetilde{A}_2)$ to T_i. This implies that G has presentation

$$G = \langle T_1, T_2, T_3 \mid T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \; mod(3), \; i = 1,2,3 \rangle$$

We shall now consider G as a subgroup of $Mod(S)$ as opposed to a subgroup of $Mod(\widetilde{S})$. There are a few cases to consider.

If S is an orientable surface containing $\widetilde{S}$ as a subsurface, and no component of $\overline{S \backslash \widetilde{S}}$ is a cylinder exterior to $\widetilde{S}$ or a disk with less than two punctures, then by corollary 1.5.3,

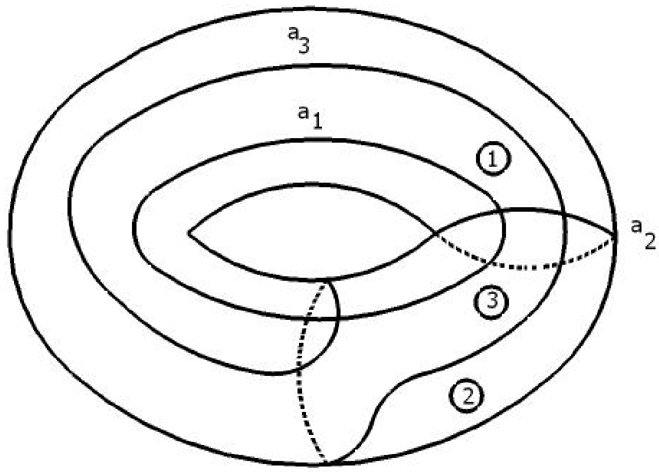


Figure 7.2: The simple closed curves a_i, $i = 1, 2, 3$ satisfy the intersection triple $(1, 1, 1)$ in $\widetilde{S}$. The surface $\widetilde{S}$, which is a closed regular neighborhood of $a_1 \cup a_2 \cup a_3$ is homeomorphic to $S_{1,3}$. Note that we numbered the boundary components to keep track of which one we cap off when studying the homomorphism $i_* : Mod(\widetilde{S}) \to Mod(S)$. It is easy to check that 3 is the distinguished boundary.

$i_* : Mod(\widetilde{S}) \to Mod(S)$ is injective. As such, the subgroup G of $Mod(S)$ generated by T_1, T_2 and T_3 is isomorphic to $\mathcal{A}(\widetilde{A}_2)$ via the geometric homomorphism mapping γ_i to T_i.

We shall now examine the cases when $i_* : Mod(\widetilde{S}) \to Mod(S)$ is not injective. First note that by theorem 4.3.4, $\widetilde{S}$ is homeomorphic to $S_{1,3}$. Moreover, it follows from theorem 5.5.4 that a_1, a_2, and a_3 may be chosen as in Figure 7.2.

The homomorphism $i_* : Mod(\widetilde{S}) \to Mod(S)$ is injective except when S is obtained from $\widetilde{S}$ by

- Capping off one boundary component with a disk.
- Capping off two boundary components with disks.
- Capping off three boundary components with disks.
- Attaching an cylinder exterior to $\widetilde{S}$.
- Attaching an exterior cylinder and capping off the remaining boundary with a disk.

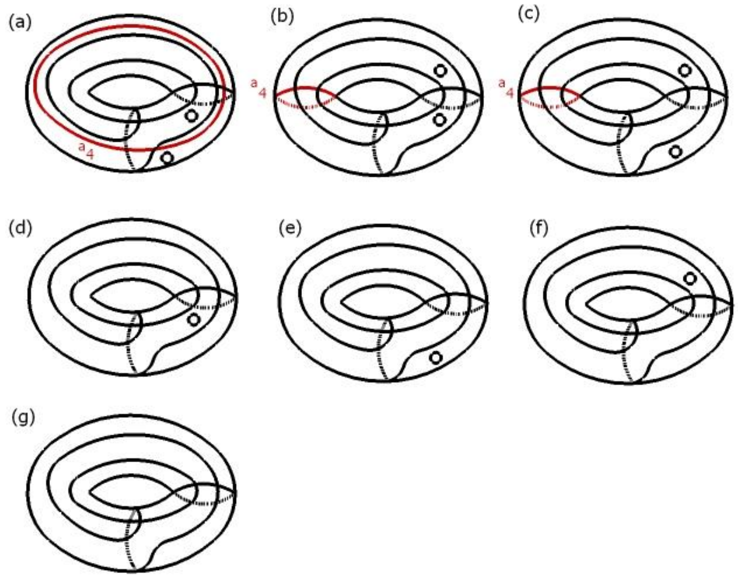


Figure 7.3: The surfaces obtained from $\widetilde{S}$ by capping off one, two and three boundary components with disks. The red curves in (a), (b), and (c) are denoted by a_4. They are introduced in each case to determine the structure of G.

First, we study the cases where S is obtained from $\widetilde{S}$ by capping off one boundary component with a disk. See figure 7.3 (a),(b), and (c) for illustration.

Suppose S is obtained from $\widetilde{S}$ by capping off ∂_1 with a disk. Introduce the simple closed curve a_4 shown in red in figure 7.3 (a) and note that $a_4 = T_2^{-1}(a_3)$. Then $T_4 = T_2^{-1}T_3T_2$. This implies that T_1, T_2 and T_4 generate G. Since a_2, a_1 and a_4 form a chain in S and since S is homeomorphic to a closed regular neighborhood of $a_2 \cup a_1 \cup a_4$, it follows by theorem 5.1.2 that G is isomorphic to $\mathcal{B}_4$.

Suppose S is obtained from $\widetilde{S}$ by capping off ∂_2 with a disk. Introduce the simple closed curve a_4 shown in figure 7.3 (b) and note that $a_4 = T_1(a_3)$. As such, $T_4 = T_1T_3T_1^{-1}$ and so G is generated by T_2, T_1 and T_4. Since a_2, a_1, and a_4 form a chain in S, and S is homeomorphic to a closed regular neighborhood of $a_2 \cup a_1 \cup a_4$, it follows from theorem 1.5.2 that G is isomorphic to $\mathcal{B}_4$.

Finally, suppose that S is obtained from $\widetilde{S}$ by capping off ∂_3 with a disk. Introduce the simple closed curve as in figure 7.3 (c) and note that S is homeomorphic to a closed regular neighborhood of $a_4 \cup a_1 \cup a_2$. Also note that $a_3 = T_4^2 T_1(a_2)$. This is the same construction as in theorem 5.5.1. By this theorem, $G \cong \mathcal{A}(\widetilde{A}_2)$.

We now study the surfaces S obtained from $\widetilde{S}$ by capping off two boundary components with disks. Refer to figure 7.3 (d), (e), and (f).

Let S be the surface obtained from $\widetilde{S}$ by capping off ∂_1 and ∂_2 with disks. It is easy to check $a_3 = T_2(a_1)$ so that $T_3 = T_2 T_1 T_2^{-1}$. As such, G is generated by T_1 and T_2 in $Mod(S)$. Since $i(a_1, a_2) = 1$, it follows by theorem 7.2.2 that G is isomorphic to $\mathcal{B}_3$.

If S is the surface obtained from $\widetilde{S}$ by capping off ∂_1 and ∂_3 with disks, note that $a_3 = T_2(a_1)$. Then $T_3 = T_2 T_1 T_2^{-1}$ and as in the preceding paragraph, G is isomorphic to $\mathcal{B}_3$.

Finally, suppose that S is obtained from $\widetilde{S}$ by capping off ∂_2 and ∂_3 with disks. In this case, it is still true that $a_3 = T_2(a_1)$. As such, G is isomorphic to $\mathcal{B}_3$.

If S be the surface obtained from $\widetilde{S}$ by capping off all three boundary components with disks (figure 7.3 (g)), then $a_3 = T_2(a_1)$ is true. So G is generated by T_1 and T_2 in $Mod(S)$. Since $i(a_1, a_2) = 1$ and S is homeomorphic to $S_{1,0}$, it follows by theorem 7.2.2 that G is isomorphic to $SL_2(\mathbb{Z})$.

We now study the surfaces S obtained from $\widetilde{S}$ by attaching an exterior cylinder c_{ij}, $(i,j) \in \{(1,2),(1,3),(2,3)\}$.

Let S be the surface obtained from $\widetilde{S}$ by attaching the exterior cylinder c_{ij}, $(i,j) \in \{(1,2),(1,3),(2,3)\}$. By theorem 1.5.2, the kernel of the homomorphism $i_* : Mod(\widetilde{S}) \to Mod(S)$ is generated by $T_{\partial_i} T_{\partial_j}^{-1}$, which is central in $Mod(\widetilde{S})$ by theorem 1.6.1. If $w \in G$ is such that $i_*(w) = 1$, then $w \in Ker(i_*)$. As such, $w = (T_{\partial_i} T_{\partial_j}^{-1})^p$ for some integer p. So, w is central in $Mod(\widetilde{S})$ and consequently $w \in Z(G)$. But $G \cong \mathcal{A}(\widetilde{A}_2)$, and so the center $Z(G)$ is trivial by theorem 7.7.1. Therefore, $w = 1$. This shows that $i_*|_G$ is injective and thus the subgroup of $Mod(S)$ generated by T_1, T_2 and T_3 is isomorphic to $\mathcal{A}(\widetilde{A}_2)$.

Finally, we study the surfaces S obtained from $\widetilde{S}$ by attaching an exterior cylinder and capping off the remaining boundary with a disk.

Let S be the surface obtained from $\widetilde{S}$ by attaching the exterior cylinder c_{12} and capping off ∂_3 with a disk. We saw that after capping off ∂_3 with a disk, the subgroup of $Mod(S_{1,2})$ generated by T_1, T_2, and T_3 is isomorphic to $\mathcal{A}(\widetilde{A}_2)$. Now attach the cylinder c_{12} to $S_{1,2}$ to obtain $S_{2,0}$. The kernel of $i_* : Mod(S_{1,2}) \to Mod(S_{2,0})$ is generated by $T_{\partial_1}T_{\partial_2^{-1}}$. Since the center of $\mathcal{A}(\widetilde{A}_2)$ is trivial, a similar argument as before shows that the subgroup G of $Mod(S_{2,0})$ generated by T_1, T_2, and T_3 is isomorphic to $\mathcal{A}(\widetilde{A}_2)$.

Let S be the surface obtained from $\widetilde{S}$ by attaching the exterior cylinder c_{13} and capping off ∂_2 with a disk. We saw that after capping off ∂_2 with a disk, the subgroup of $Mod(S_{1,2})$ generated by T_1, T_2, and T_3 is isomorphic to $\mathcal{B}_4$. Now attach the cylinder c_{13} to $S_{1,2}$ to get $S_{2,0}$. The kernel of $i_* : Mod(S_{1,2}) \to Mod(S_{2,0})$ is generated by $T_{\partial_1}T_{\partial_3^{-1}}$, which is central in $Mod(S_{1,2})$ by theorem 1.6.1. If $i_*(w) = 1$ for some $w \in G < Mod(S_{1,2})$, then $w \in Ker(i_*)$. Thus, $w \overset{(1)}{=} (T_{\partial_1}T_{\partial_3}^{-1})^p$ for some integer p. So w lies in the center of $Mod(S_{1,2})$, and in particular, w is in the center of $G \cong \mathcal{B}_4$. By theorem 2.2.1 and theorem 5.1.2, the center of G is infinite cyclic generated by $(T_4T_1T_2)^4$. Moreover, $(T_4T_1T_2)^4 = T_{\partial_1}T_{\partial_3}$ by the chain relation. So $w \overset{(2)}{=} (T_{\partial_1}T_{\partial_3})^q$ for some integer q. By (1) and (2), and the fact that $[T_{\partial_1}, T_{\partial_3}] = 1$, we have

$$\begin{aligned} T_{\partial_1}^p T_{\partial_3}^{-p} &= T_{\partial_1}^q T_{\partial_3}^q \Leftrightarrow \\ T_{\partial_1}^{p-q} &= T_{\partial_3}^{p+q} \end{aligned}$$

Since $\partial_1 \neq \partial_3$, corollary 1.3.10 implies $p - q = p + q = 0$. This implies that $p = q = 0$. As such, $w = 1$ and so $i_*|_G$ is injective. Therefore, the subgroup G of $Mod(S_{2,0})$ generated by T_1, T_2, and T_3 is isomorphic to $\mathcal{B}_4$.

Finally, let S be the surface obtained from $\widetilde{S}$ by attaching the exterior cylinder c_{23} and capping off ∂_1 with a disk. We saw that after capping off ∂_1 with a disk, the subgroup G of $Mod(S_{1,2})$ generated by T_1, T_2, and T_3 is isomorphic to $\mathcal{B}_4$. Now attach the cylinder c_{13}

to $S_{1,2}$ to get $S_{2,0}$. A similar argument to the one in the preceding case shows that G is isomorphic to $\mathcal{B}_4$. □

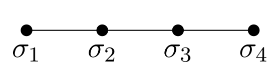
7.8 The case $(2, 1, 0)$

Assume that a_1, a_2, and a_3 satisfy the triple $(2, 1, 0)$ in S. There are two cases to investigate, depending on whether the algebraic intersection number $\hat{i}(a_1, a_2)$ is ± 2 or 0. In the following subsections, we study the subgroups of $Mod(S)$ generated by Dehn twists along curves satisfying the configurations $(2_{\pm 2}, 1, 0)$ and $(2_0, 1, 0)$.

7.8.1 $(2_{\pm 2}, 1, 0)$

Suppose that a_1, a_2, and a_3 satisfy the triple $(2_{\pm 2}, 1, 0)$ in S, and consider the subsurface $N_\epsilon \subset S$, which is a closed regular neighborhood of $a_1 \cup a_2 \cup a_3$. N_ϵ is homeomorphic to $S_{2,1}$. This can be seen as follows. By tracing ∂F (see figure 7.4) , it is clear that N_ϵ has one boundary component. Moreover, N_ϵ deformation retracts to the graph Γ consisting of $a_1 \cup a_2 \cup a_3$. Since the Euler characteristic is homotopy type invariant, it follows that $\chi(N_\epsilon) = \chi(\Gamma)$. Hence, $\chi(N_\epsilon) = 2 - 2g - 1 = -3 = v - e = \chi(\Gamma)$ implies $g_{N_\epsilon} = 2$. So, N_ϵ is homeomorphic to $S_{2,1}$.

Proposition 7.8.1. *Let a_1, a_2, and a_3 be isotopy classes of simple closed curves satisfying the triple $(2_{\pm 2}, 1, 0)$ in S, and denote by G be the subgroup of $Mod(S)$ generated by the Dehn twists T_1, T_2, and T_3. Consider the five strand braid group $\mathcal{B}_5$ whose presentation is encoded in the Coxeter graph*

$A_n =$ 

Assume that $S \neq S_{2,0}$. Then G is isomorphic to the subgroup of $\mathcal{B}_5$ generated by $\sigma_2^2 \sigma_3 \sigma_2^{-2}$, σ_1, and σ_4. Moreover, G is finitely presented and is isomorphic to an infinite index subgroup of $\mathcal{B}_5$.

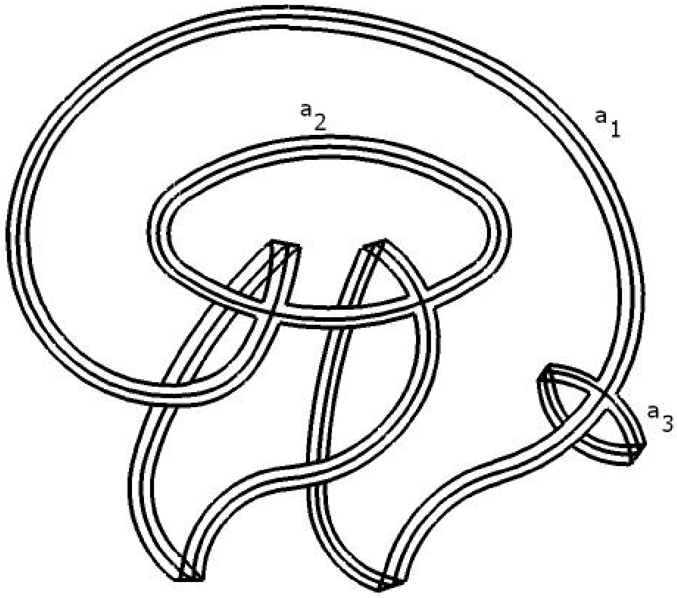


Figure 7.4: The surface N_ϵ formed by a closed regular neighborhood of $a_1 \cup a_2 \cup a_3$, where the a_i satisfy $(2_{\pm 2}, 1, 0)$.

Proof. Consider the subsurface N_ϵ of S which is a closed regular neighborhood of $a_1 \cup a_2 \cup a_3$. As shown above N_ϵ is homeomorphic to $S_{2,1}$, and the positions of the a_i in F can be seen in figure 7.5. Introduce the curves a_4 and a_5 as in figure 7.6, and denote by G_1 the subgroup of $Mod(N_\epsilon)$ generated by T_2, T_3, T_4, and T_5. By theorem 5.1.2, G_1 is isomorphic to $\mathcal{B}_5$. That is, G_1 has generators T_2, T_3, T_4 and T_5, and defining relations

$$\begin{aligned} T_2T_3 &= T_3T_2 && (1) \\ T_2T_4T_2 &= T_4T_2T_4 && (2) \\ T_2T_5 &= T_5T_2 && (3) \\ T_3T_4 &= T_4T_3 && (4) \\ T_3T_5T_3 &= T_5T_3T_5 && (5) \\ T_4T_5T_4 &= T_5T_4T_5 && (6) \end{aligned}$$

It is not hard to see that $a_1 = T_4^2(a_5)$. By fact 1.3.3, $T_1 = T_4^2 T_5 T_4^{-2}$. This implies that G

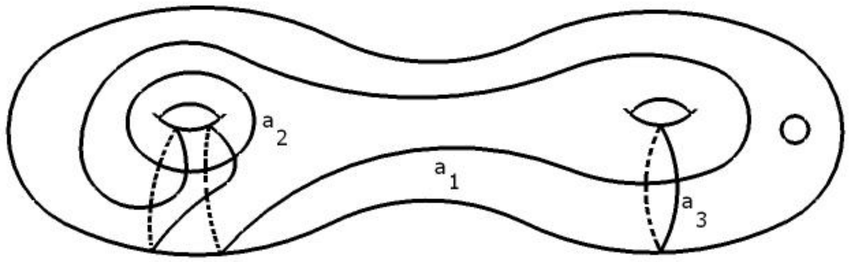


Figure 7.5: The positions of a_1, a_2, and a_3 in N_ϵ.

is a subgroup of G_1. It is precisely the subgroup of G_1 generated by $T_4^2 T_5 T_4^{-2}$, T_2, and T_3. The restriction of the following homomorphism to G establishes the desired isomorphism

$$\phi : G_1 \to \mathcal{B}_5$$

$$T_2 \mapsto \sigma_1$$

$$T_4 \mapsto \sigma_2$$

$$T_5 \mapsto \sigma_3$$

$$T_3 \mapsto \sigma_4$$

Finally, the following Magma code establishes that G is finitely presented and is isomorphic to a subgroup of infinite index in $\mathcal{B}_5$.

```
F<a,b,c,d> := Group<a, b, c, d | a*b*a = b*a*b, a*c = c*a, a*d = d*a,
b*c*b = c*b*c, b*d = d*b, c*d*c = d*c*d >;
G<A,B,C> := sub<F | a, d, b^2*c*b^-2 >;
Index(F, sub<F | a, d, b^2*c*b^-2 >: CosetLimit:=10^8,Hard:=true,
Mendelsohn:=true);
G;


0
Finitely presented group G on 3 generators
Generators as words in group F
```

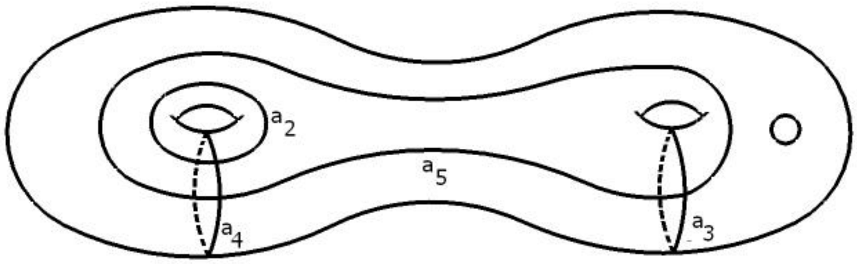


Figure 7.6: The curves a_2, a_3, a_4, and a_5 form a chain in N_ϵ.

```
A = a
B = d
C = b^2 * c * b^-2
```

In the above code, a, b, c, and d represent σ_1, σ_2, σ_3, and σ_4 respectively. As shown in the output, G is a finitely presented group. Moreover, the number 0 in the output means $[F : G] = \infty$. $\square$

We remark that the reason for omitting $S = S_{2,0}$ in proposition 7.8.1 is because $i_* : Mod(S_{2,1}) \to Mod(S_{2,0})$ is not injective. Consequently, the restriction of i_* to the subgroup G of $Mod(S_{2,1})$ might not be injective as well. Now, although $Ker(i_*)$ can be computed, it is relatively complicated. This makes determining $Ker(i_*|_G)$ (which equals $Ker(i_*) \cap G$) a nontrivial task.

7.8.2 $(2_0, 1, 0)$

Suppose that a_1, a_2, and a_3 satisfy the intersection triple $(2_0, 1, 0)$ in S, and consider the subsurface $N_\epsilon \subset S$ which is a closed regular neighborhood of $a_1 \cup a_2 \cup a_3$. N_ϵ is homeomorphic to $S_{1,3}$. To see that, note that N_ϵ has three boundary components. This can be checked by tracing the boundary in Figure 7.7 (left). Since N_ϵ deformation retracts to the graph Γ which is $a_1 \cup a_2 \cup a_3$. Clearly, Γ has $v = 3$ vertices and $e = 6$ edges. Since

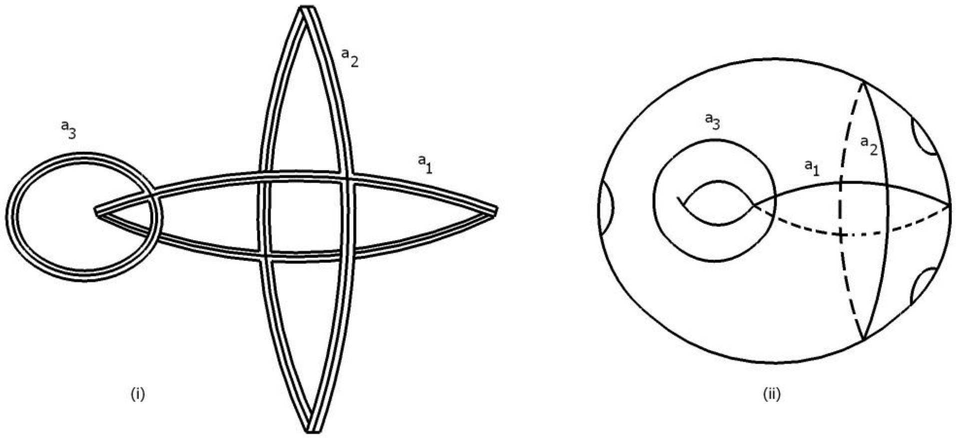


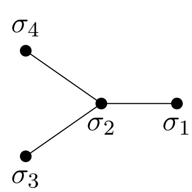
Figure 7.7: N_ϵ is a closed regular neighborhood of $a_1 \cup a_2 \cup a_3$, where the a_i satisfy $(2_0, 1, 0)$.

the Euler characteristic is invariant under homotopy equivalence, it follows that

$$\chi(N_\epsilon) = 2 - 2g - 3 = 3 - 6 = v - e = \chi(\Gamma)$$

As such, N_ϵ is homeomorphic to $S_{1,3}$ (see figure 7.7).

Proposition 7.8.2. *Suppose that a_1, a_2 and a_3 satisfy the triple $(2_0, 1, 0)$ and denote by G the subgroup of $Mod(N_\epsilon)$ generated by T_i, $i = 1, 2, 3$. Consider the Artin group $\mathcal{A}(D_4)$ whose presentation is encoded in the Coxeter graph*

$D_4 =$ 

Set $\langle \sigma_5 \rangle \cong \mathbb{Z}$. Then G is isomorphic to the subgroup of $\mathcal{A}(D_4) \times \mathbb{Z}$ generated by σ_1, $\sigma_5(\sigma_4\sigma_2\sigma_3)^4$ and σ_2. Moreover, this subgroup is finitely presented and has infinite index in $\mathcal{A}(D_4) \times \mathbb{Z}$.

Proof. Introduce the curves a_4, a_5 and a_6 as in figure 7.8. It follows from the chain relation that $T_2 = T_6^{-1}(T_5T_3T_4)^4$. Denote by G_1 the subgroup of $Mod(N_\epsilon)$ generated by T_1, T_3, T_4,

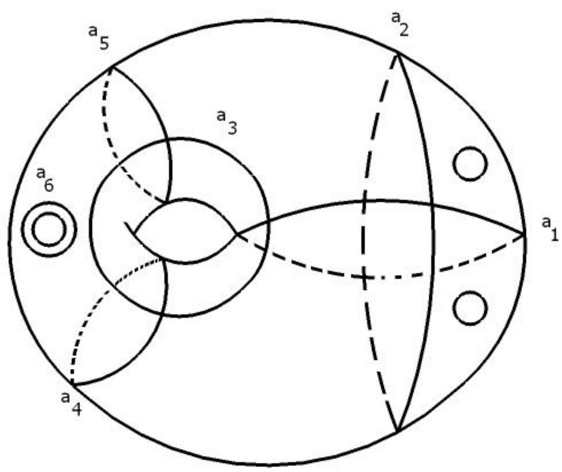


Figure 7.8: This picture depicts the surface N_ϵ which is a closed regular neighborhood of $a_1 \cup a_2 \cup a_3$.

T_5 and T_6. By theorem 5.1.2, the subgroup of $Mod(N_\epsilon)$ generated by T_1, T_3, T_4, and T_5 is isomorphic to $\mathcal{A}(D_4)$. Since T_6 is central in $Mod(N_\epsilon)$, it follows that $G_1 \cong \mathcal{A}(D_4) \times \mathbb{Z}$. Now it is easy to check that the following homomorphism restricted to G is in fact the desired isomorphism

$$\phi : G_1 \to \mathcal{A}(D_4) \times \mathbb{Z}$$

$$T_1 \mapsto \sigma_1$$

$$T_3 \mapsto \sigma_2$$

$$T_4 \mapsto \sigma_3$$

$$T_5 \mapsto \sigma_4$$

$$T_6 \mapsto \sigma_5$$

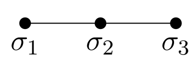
That G is finitely presented and is isomorphic to an infinite index subgroup of $\mathcal{A}(D_4) \times \mathbb{Z}$ is revealed in the following Magma code.

```
F<a,b,c,d,e> := Group<a, b, c, d, e | a*b*a = b*a*b, a*c = c*a,
a*d = d*a, a*e = e*a, b*c*b = c*b*c, b*d*b = d*b*d, b*e = e*b,
c*d*c = d*c*d, c*e = e*c, d*e = e*d >;
```

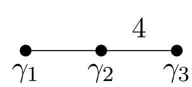
```
G<A,B,C> := sub<F | a, e^-1*(c*b*d)^4, c >;

Index(F, sub<F | a, e^-1*(c*b*d)^4, c >:


CosetLimit:=2*10^6,Hard:=true,Mendelsohn:=true);

G;


0

Finitely presented group G on 3 generators

Generators as words in group F

    A = a

    B = e^-1 * c * b * d * c * b * d * c * b * d * c * b * d

    C = c
```

In the above code, a, b, c, d, and e represent σ_i for $i = 1, 2, 3, 4$ and 5 respectively. □

Proposition 7.8.3. *Let $\hat{N}_\epsilon$ be the surface obtained from N_ϵ (from proposition 7.8.2) by capping off the boundary component parallel to a_6 with a disk. Consider a_1, a_2, and a_3 in $\hat{N}_\epsilon$ and let G be the subgroup of $Mod(\hat{N}_\epsilon)$ generated by T_i, $i = 1, 2, 3$. Consider the four strand braid group $\mathcal{B}_4$ whose presentation is encoded in the Coxeter group*

$A_3 =$ 

Then G is isomorphic to the subgroup of $\mathcal{B}_4$ generated by σ_3, $(\sigma_1\sigma_2)^6$, and σ_2. Moreover, this subgroup is finitely presented and has infinite index in $\mathcal{B}_4$. Further, consider the Artin group $\mathcal{A}(B_3)$, whose presentation is encoded in the Coxeter graph

$B_3 =$ 

Then G is isomorphic to the subgroup of $\mathcal{A}(B_3)$ generated by γ_2, γ_3^2, and γ_1. Moreover, this subgroup has infinite index in $\mathcal{A}(B_3)$.

Proof. After capping off the boundary parallel to a_6 with a disk, a_4 becomes isotopic to a_5. It follows from theorem 5.1.2 that the subgroup G_1 of $Mod(\hat{N}_\epsilon)$ generated by T_1, T_3 and T_4 is isomorphic to $\mathcal{B}_4$. By the chain relation implies $T_2 = (T_3T_4)^6$. As such, G is a subgroup of G_1. Precisely, $G \leq G_1$ is generated by T_1, $(T_3T_4)^6$, and T_3. The following Magma code shows that G is finitely presented and $[\mathcal{B}_4 : G] = \infty$.

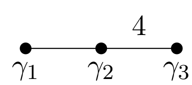
```
F<a,b,c> := Group<a,b,c | a*b*a = b*a*b, a*c = c*a, b*c*b = c*b*c >;
G<A,B,C> := sub<F | c, (a*b)^6, b >;
Index(F, sub<F | c, (a*b)^6, b >: CosetLimit:=2*10^6,Hard:=true,
Mendelsohn:=true);
G;


0
Finitely presented group G on 3 generators
Generators as words in group F
    A = c
    B = (a * b)^6
    C = b
```

Let G_2 be the group generated by T_1, $(T_3T_4)^3$, and T_3. Then G is a subgroup of G_2. The following Magma code reveals that G_2 is isomorphic to $\mathcal{A}(B_3)$ and that $[\mathcal{B}_4 : \mathcal{A}(B_3)] = 8$.

```
F<a,b,c> := Group<a,b,c | a*b*a = b*a*b, a*c = c*a, b*c*b = c*b*c >;
G<x,y,z> := sub<F | b, (a*b)^3, c >;
Index(F,G);
Rewrite(F,~G);
G;


8
Finitely presented group G on 3 generators
```

```
Index in group F is 8 = 2^3

Generators as words in group F

    x = b

    y = (a * b)^3

    z = c

Relations

    (x^-1, y) = Id(G)

    z * x * z^-1 * x^-1 * z^-1 * x = Id(G)

    y^-1 * z * y * z * y * z^-1 * y^-1 * z^-1 = Id(G)
```

In the above code, a, b, and c represent T_4, T_3, and T_1 respectively. Set $\gamma_1 = T_3$, $\gamma_2 = T_1$, and $\gamma_3 = (T_4T_3)^3$. As seen above, $G_2 \cong \mathcal{A}(B_3)$ and its presentation is encoded in the Coxeter graph

$B_3 =$ 

G is the subgroup of G_2 generated by γ_2, γ_3^2 and γ_1. The following Magma code shows that $[\mathcal{A}(B_3) : G] = \infty$.

```
F<a,b,c> := Group<a,b,c | a*b*a = b*a*b, a*c = c*a,

b*c*b*c = c*b*c*b >;

G<A,B,C> := sub<F | b, c^2, a >;

Index(F, sub<F | b, c^2, a >: CosetLimit:=2*10^6,Hard:=true,

Mendelsohn:=true);

G;


0

Finitely presented group G on 3 generators

Generators as words in group F

    A = b
```

```
B = c^2

C = a
```

In the above code, a, b, and c represent γ_1, γ_2, and γ_3 respectively. □

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